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INTEGRAL REPRESENTATION THEOREMS VIA BANACH ALGEBRAS

by George MALTESE

1. INTRODUCTION

Many classical integral representation theorems of analysis can be obtained as special cases of the Choquet Representation Theorem [6], [7], [14] or the Krein-Milman Theorem. The procedure involves the definition of a suitable convex compact set in some locally convex space and an explicit description of the extreme points of this set. The latter is often a non-trivial task, therefore it seems appropriate to develop alternative methods which are general enough to yield a class of integral representation theorems. In many situations in which an integral representation formula is sought, there is a natural commutative Banach algebra inherent in the background. For example in the case of Bochner's theorem for positive definite functions on a locally compact abelian group G, the natural Banach algebra is the convolution algebra $L^1(G)$. In the case of the Schoenberg-Eberlein theorem for Fourier-Stieltjes transforms on locally compact abelian groups, the Banach algebra is again the convolution algebra. In the case of the Spectral Theorem for a normal operator T on a Hilbert space $\mathscr{H}$, the natural Banach algebra is the closed commutative * algebra generated by T and the identity operator.

In this paper we show that the above mentioned theorems are all special cases of a general result (Theorem 1) on the integral representation of certain linear forms defined on commutative Banach algebras. Specialization of Theorem 1 to symmetric Banach algebras yields a generalized version (Theorem 2) of a result of Raikov [10] for positive functionals on such algebras.

The proof of Theorem 1 is straight forward and its version for positive functionals on involution algebras is classical [11]. The main point here is the relative ease of application of Theorem 1 to a variety of situations.

2. Integral representation theorems for linear functionals

Let A be a commutative Banach algebra over $\mathbf{C}$ and let Δ denote the locally compact space of regular maximal ideals of A. For each $x \in A$ we use $\hat{x}$ to denote the Gelfand-transform; i.e., $\hat{x}$ is the continuous mapping from Δ to $\mathbf{C}$ defined by the relations:

$$\hat{x}(m) = m(x) \quad \text{for} \quad m \in \Delta .$$

By $C_0(\Delta)$ we shall denote the algebra of all complex-valued continuous functions on Δ which vanish at infinity. For any subset $\mathscr{A} \subset A$ we shall use the notation $\hat{\mathscr{A}}$ to denote the set $\{ \hat{x} : x \in \mathscr{A} \}$. As usual $\| \hat{x} \|_\infty$ denotes the supremum norm.

THEOREM 1. Let f be a linear form on the complex commutative Banach algebra A and let $\mathscr{A}$ be a linear subspace of A. The following two statements are equivalent:

(1) There exists a constant M such that

$$| f(x) | \leqslant M \| \hat{x} \|_\infty \quad \text{for every } x \in \mathscr{A} .$$

(2) There exists a bounded complex Radon measure μ on Δ such that

$$f(x) = \int_\Delta \hat{x}(m)\, d\mu(m) \quad \text{for every} \quad x \in \mathscr{A}.$$

Proof. The implication (2) ⇒ (1) is clear with $M = \| \mu \|$. We shall prove (1) ⇒ (2). Define a mapping $L : \hat{\mathscr{A}} \to \mathbf{C}$ by

$$L(\hat{x}) = f(x) .$$

It follows from (1) that L is well-defined, and that

$$| L(\hat{x}) | \leqslant M \| \hat{x} \|_\infty \quad \text{for every} \quad \hat{x} \in \hat{\mathscr{A}}$$

and so L is continuous with $\| L \| \leqslant M$. Using the Hahn-Banach Theorem we can extend L to a bounded linear form L_0 on $C_0(\Delta)$ and by the Riesz Representation Theorem we obtain the existence of a bounded complex Radon measure μ on Δ such that

$$\| \mu \| = \| L \| = \| L_0 \| \quad \text{and}$$

$$L_0(\varphi) = \int_\Delta \varphi(m)\, d\mu(m) \qquad \text{for every} \quad \varphi \in C_0(\Delta).$$

In particular

$$f(x) = L(\hat{x}) = L_0(\hat{x}) = \int_\Delta \hat{x}(m)\, d\mu(m) \quad \text{for every} \quad x \in \mathscr{A}.$$

Remark: Suppose that A has an identity and that $\hat{A}$ is closed under complex conjugation, then since $\hat{A}$ contains constants and separates the points of Δ, the Stone-Weierstraß Theorem implies that $\hat{A}$ is dense in $\mathbf{C}(\Delta)$, the algebra of all complex-valued continuous functions on the compact Hausdorff space Δ. If we impose these additional conditions on A and if we take $\mathscr{A} = A$ in Theorem 1, we can conclude that in this case the representing measure μ is uniquely determined.

If the algebra A has a continuous involution, one can use Theorem 1 to derive an extended version of a theorem due to Raikov [10]. We proceed to describe the situation.

Let A be a complex commutative Banach algebra with an isometric involution $*$ and a bounded approximate identity $\{ u_\lambda \}_{\lambda \in \Lambda}$ i.e., a net satisfying the following conditions:

$$\| u_\lambda \| \leqslant 1 \quad \text{for each} \quad \lambda \in \Lambda,$$
$$\| u_\lambda x - x \| \to 0 \quad \text{for each} \quad x \in A.$$

A continuous *positive* functional on A is an element $f \in A'$ such that $f(x^*x) \geqslant 0$ for every $x \in A$. If f is a continuous positive functional on A then the Cauchy-Schwarz inequality is valid (Dixmier [8, p. 23]) and this implies the following facts:

$$f(u_\lambda) \to \| f \|$$
$$| f(x) |^2 \leqslant \| f \| \, f(x^*x) \quad \text{for every} \quad x \in A.$$

If the involution is *symmetric*, which means $(x^*)^\wedge = \overline{\hat{x}}$ for every $x \in A$ or, equivalently, that every $m \in \Delta$ is a *positive* linear functional, then by modifying a classical method of Gelfand-Raikov-Silov [10; p. 62] one can prove that

$$| f(x) | \leqslant \| f \| \, \| \hat{x} \|_\infty \quad \text{for every} \quad x \in A.$$

As a corollary to Theorem 1 and the above discussion we obtain the following extended theorem of Raikov [10; p. 64], see also Bucy-Maltese [4]):

THEOREM 2. Let A be a commutative Banach algebra with an isometric involution which is symmetric. Suppose that A has a bounded approximate identity and let $f \in A'$ be a continuous positive functional. Then there exists a unique positive Radon measure μ on Δ such that $\| \mu \| = \| f \|$ and

$$f(x) = \int_{\Delta} \hat{x}(m) \, d\mu(m) \quad \text{for every} \quad x \in A .$$

Proof. From the above remarks we know that

$$| f(x) | \leqslant \| f \| \, \| \hat{x} \|_{\infty} \quad \text{for every} \quad x \in A .$$

By Theorem 1 there exists a complex Radon measure μ on Δ such that $\| \mu \| \leqslant \| f \|$ and

$$f(x) = \int_{\Delta} \hat{x}(m) \, d\mu(m) \quad \text{for every} \quad x \in A .$$

This formula implies

$$| f(x) | \leqslant \| \hat{x} \|_{\infty} \, \| \mu \| \leqslant \| x \| \, \| \mu \| \quad \text{for every} \quad x \in A ,$$

so that $\| f \| \leqslant \| \mu \|$ and hence $\| f \| = \| \mu \|$.

Since $\hat{A}$ is a self-adjoint subalgebra of $\mathbf{C}_0(\Delta)$ which separates points and for each $m \in \Delta$ contains a function $\hat{x}$ such that $\hat{x}(m) \neq 0$ (in fact [1]) there exists an element u_β of the approximate identity such that $\hat{u}_\beta(m) \neq 0$), the Stone-Weierstraß Theorem implies the uniqueness of the measure μ. The positivity of μ also follows from the fact that $\hat{A}$ is dense in $\mathbf{C}_0(\Delta)$. In fact if p is a non-negative function in $\mathbf{C}_0(\Delta)$, then $p = | q |^2$ for some $q \in \mathbf{C}_0(\Delta)$. Choose a sequence $\{ x_n \}$ in A such that

$$\hat{x}_n \to q .$$

[1]) If $m \in \Delta$, then $\| m \| \neq 0$ and by the assumption of symmetry m is a positive functional. Therefore, as mentioned above, $\| m \| = \lim_{\alpha} m(u_\alpha)$ so that there must exist some u_β of the approximate identity such that $m(u_\beta) \neq 0$.

Since $(x_n^*)^\wedge = \bar{\hat{x}}_n$ it follows that $(x_n^*)^\wedge \to \bar{q}$ and hence

$$(x_n x_n^*)^\wedge \to |q|^2 = p .$$

This implies

$$\begin{aligned} \int_\Delta p(m)\, d\mu(m) &= \lim_n \int_\Delta (x_n^* x_n)^\wedge)(m)\, d\mu(m) \\ &= \lim_n f(x_n^* x_n) \geqslant 0 , \end{aligned}$$

so that μ is a positive measure and this completes the proof.

If A has an identity, as is the case in Raikov's original version, the above proof can be somewhat simplified.

THEOREM 3 (Raikov). Let A be a complex commutative Banach algebra with an identity e and with an isometric involution which is symmetric. If f is a continuous positive functional on A, then there exists a unique positive Radon measure μ on Δ such that $\|\mu\| = \|f\|$ and

$$f(x) = \int_\Delta \hat{x}(m)\, d\mu(m) \quad \text{for every} \quad x \in A .$$

Proof. As above we know that

$$|f(x)| \leqslant \|f\| \, \|\hat{x}\|_\infty \quad \text{for every} \quad x \in A .$$

From Theorem 1 there exists a complex Radon measure μ on Δ such that $\|\mu\| \leqslant \|f\|$ and

$$f(x) = \int_\Delta \hat{x}(m)\, d\mu(m) \quad \text{for every} \quad x \in A .$$

Hence $\|\mu\| \leqslant \|f\| = f(e) = \mu(1) \leqslant \|\mu\|$ so that $\mu(1) = \|\mu\|$ which is enough to imply that μ is positive. The uniqueness of μ follows as in the Remark to Theorem 1.

3. Applications of the integral representation theorems

Application 1 (Bochner's Theorem). Let G be a locally compact abelian group and let $\hat{G}$ denote the (locally compact) character group. Denote

Haar measure on G by dt and the group algebra (with convolution as multiplication) by $L^1(G)$.

Definition: A function $p \in L^\infty(G)$ is said to be *positive definite* provided

$$\int_G \int_G F(t)\overline{F(s)}\, p(t-s)\, dt ds \geqslant 0$$

for every $F \in L^1(G)$.

Using the natural involution $F \to \tilde{F}$ of $L^1(G)$ defined by $\tilde{F}(t) = \overline{F(-t)}$, we can rewrite the definition of positive definiteness as follows: $p \in L^\infty(G)$ is positive definite provided

$$\int_G F*\tilde{F}(t)\, p(t)\, dt \geqslant 0 \quad \text{for all} \quad F \in L^1(G).$$

For a positive definite function p define the mapping $P: L^1(G) \to \mathbf{C}$ as follows:

$$P(F) = \int_G F(t)\, p(t)\, dt.$$

Therefore P is a continuous positive linear functional on the symmetric involution algebra $L^1(G)$ such that $\|P\| = \|p\|_\infty$. By the discussion preceding Theorem 2 and the fact that $L^1(G)$ has an approximate identity, we know that

$$|P(F)| \leqslant \|P\| \; \|\hat{F}\|_\infty \quad \text{for every} \quad F \in L^1(G),$$

where $\hat{F}$ is the Fourier-Gelfand transform. By Theorem 2 there exists a unique positive Radon measure μ on $\Delta(L^1(G))$ such that

$$P(F) = \int_{\Delta(L^1(G))} \hat{F}(\Gamma)\, d\mu(\Gamma) \quad \text{for every} \quad F \in L^1(G).$$

It is classical that $\Delta\left(L^1(G)\right)$ is homeomorphic to the locally compact character group $\hat{G}$ under the correspondence $\Gamma \leftrightarrow \gamma$ where

$$\Gamma(F) = \hat{F}(\gamma) \quad \text{for every} \quad F \in L^1(G).$$

Therefore using the same symbol for the measure induced on $\hat{G}$ by μ under this identification, we obtain

$$P(F) = \int_{\hat{G}} \hat{F}(\gamma)\, d\mu(\gamma) \quad \text{for every} \quad F \in L^1(G)$$

which implies

$$\int_G F(t)\, p(t)\, dt = \int_{\hat{G}} d\mu(\gamma) \int_G F(t)\, \overline{\gamma(t)}\, dt$$
$$= \int_G F(t) \int_{\hat{G}} \overline{\gamma(t)}\, d\mu(\gamma)\, dt\,.$$

Since this holds for all $F \in L^1(G)$ we conclude that

$$p(t) = \int_{\hat{G}} \overline{\gamma(t)}\, d\mu(\gamma) \quad \text{for almost all } t\,.$$

This is, of course, the famous Bochner characterization of positive definite functions ([5], [6], [7], [10], [11], [16]).

Application 2 (A theorem of Schoenberg-Eberlein). A complex-valued function ψ defined on the locally compact abelian group G is called a *Fourier-Stieltjes transform* if there exists a bounded Radon measure v_ψ defined on the dual group $\hat{G}$ such that

$$\psi(t) = \int_{\hat{G}} \overline{\gamma(t)}\, dv_\psi(\gamma) \quad \text{for almost all} \quad t \in G\,.$$

Definition: A measurable complex-valued function ψ on G satisfies the *Schoenberg condition* provided

(a) ψ is integrable on every compact set;

(b) there exists a constant M such that

$$\left| \int_G F(t)\, \psi(t)\, dt \right| \leqslant M \sup_\gamma \left| \int_G F(t)\, \overline{\gamma(t)}\, dt \right|$$

for every $F \in K(G)$, where $K(G)$ is the set of continuous functions on G with compact support.

The following theorem is due to Schoenberg [17] for the case $G = \mathbf{R}$ and to Eberlein [9] in the general case.

THEOREM. A measurable complex-valued ψ has a representation as a Fourier-Stieltjes transform if and only if ψ satisfies the Schoenberg condition.

Proof. It is immediate that if ψ is a Fourier-Stieltjes transform then ψ satisfies the Schoenberg condition for the constant $M = \| v_\psi \|$.

If now ψ satisfies the Schoenberg condition define $L : L^1(G) \to \mathbf{C}$ as usual by

$$L(F) = \int_G F(t)\, \psi(t)\, dt\,.$$

By hypothesis

$$|L(F)| \leqslant M \|\hat{F}\|_\infty \quad \text{for every} \quad F \in K(G).$$

From Theorem 1 there exists a bounded Radon measure v_ψ defined on $\hat{G}$ such that

$$L(F) = \int_{\hat{G}} \hat{F}(\gamma)\, dv_\psi(\gamma) \quad \text{for} \quad F \in K(G).$$

Therefore it follows that

$$\int_G F(t)\,\psi(t)\,dt = \int_{\hat{G}} dv_\psi(\gamma) \int_G F(t)\,\overline{\gamma(t)}\,dt \quad \text{for all} \quad F \in K(G)$$

and hence

$$\psi(t) = \int_{\hat{G}} \overline{\gamma(t)}\, dv_\psi(\gamma) \quad \text{for almost all} \quad t \in G,$$

which completes the proof.

Application 3 (Positive definite functions on abelian semigroups). Let S be an abelian semigroup, i.e. a set equipped with a composition law denoted by $+$ such that the commutative and associative laws are valid. We shall assume the existence of a neutral element 0. By $l^1(S)$ we shall denote the real commutative Banach algebra of functions $f: S \to \mathbf{R}$ with the property that

$$\|f\| = \sum_{a \in S} |f(a)| < \infty.$$

Multiplication in $l^1(S)$ is defined by convolution; viz.,

$$f * g(a) = \sum_{\substack{s,\, t \in S \\ s+t=a}} f(s)\, g(t).$$

By a *character* on S we shall mean a function $\gamma: S \to [-1, 1]$ which satisfies

(i) $\gamma(0) = 1$

(ii) $\gamma(s+t) = \gamma(s)\,\gamma(t)$ for all $s, t \in S$.

The set $\hat{S}$ of all characters is an abelian semigroup under pointwise multiplication. If we endow $\hat{S}$ with the topology of pointwise convergence, then $\hat{S}$ is a compact Hausdorff space. By $\Delta(l^1(S))$ we shall denote the compact subset of $l^1(S)'$ consisting of all continuous homomorphisms from $l^1(S)$ onto $\mathbf{R}$ endowed with the weak * topology.

If $\gamma \in \hat{S}$ then the mapping

$$\Gamma_\gamma : l^1(S) \to \mathbf{R}$$

defined by the relations

$$\Gamma_\gamma(f) = \sum_{s \in S} f(s)\gamma(s) \quad \text{for} \quad f \in l^1(S)$$

is a non-trivial continuous homorphism from $l^1(S)$ onto $\mathbf{R}$, so that $\Gamma_\gamma \in \Delta(l^1(S))$. Conversely for every $\Gamma \in \Delta(S))$ there exists $\gamma \in \hat{S}$ with

$$\Gamma(f) = \sum_{s \in S} f(s)\gamma(s) \quad \text{for} \quad f \in l^1(S).$$

It is easily verified that the mapping

$$\gamma \to \Gamma_\gamma$$

is a homeomorphism of $\hat{S}$ onto $\Delta(l^1(S))$. We shall identify $\hat{S}$ with $\Delta(l^1(S))$ via this homeomorphism and consider the Gelfand transform $\hat{f}$ of $f \in l^1(S)$ as a continuous function on $\hat{S}$ via

$$\hat{f}(\gamma) = \hat{f}(\Gamma_\gamma).$$

A bounded function $\varphi : S \to \mathbf{R}$ is called *positive definite* if

$$\sum_{s \in S} f * f(s)\varphi(s) \geqslant 0 \quad \text{for all} \quad f \in l^1(S).$$

In the sequel we shall *assume* that the spectral radius formula is valid in the real Banach algebra $l^1(S)$; i.e., we shall assume that

$$\|\hat{f}\|_\infty = \lim_{n \to \infty} \|f^n\|^{\frac{1}{n}}$$

for every $f \in l^1(S)$. One can easily prove that the spectral radius formula is true for the simple functions ε_a, $\varepsilon_a + \varepsilon_b$, $\lambda\varepsilon_a$ and $\varepsilon_a * \varepsilon_b$, where for each $a \in S$ the function ε_a is defined by the relations

$$\varepsilon_a(s) = \begin{cases} 0 & \text{for} \quad s \neq a \\ 1 & \text{for} \quad s = a. \end{cases}$$

In the general context of real Banach algebras, of course, the spectral radius formula need not hold.

The following theorem was demonstrated *without* the assumption of the spectral radius formula for $l^1(S)$. The theorem was obtained by Berg-Christensen-Ressel [3] using the Krein-Milman Theorem along with an interesting characterization of the extreme points of the convex compact set of normalized positive definite functions.

THEOREM. Assume that the spectral radius formula is valid for each $f \in l^1(S)$. If φ is a positive definite function on S, then there exists a unique positive Radon measure μ on $\hat{S}$ such that

$$\varphi(s) = \int_{\hat{S}} \gamma(s)\, d\mu(\gamma) \quad \text{for every} \quad s \in S.$$

Proof. The functional $L : l^1(S) \to \mathbf{R}$ defined by

$$L(f) = \sum_{s \in S} f(s)\varphi(s)$$

is positive, i.e. $L(f * f) \geqslant 0$. By our assumption of the validity of the spectral radius formula we have

$$|L(f)| \leqslant \varphi(0) \|\hat{f}\|_\infty \quad \text{for every} \quad f \in l^1(S).$$

Exactly as in Theorem 1 and Theorem 3 there exists a positive Radon measure μ on $\hat{S}$ such that

$$L(f) = \int_{\hat{S}} \hat{f}(\gamma)\, d\mu(\gamma) \quad \text{for every} \quad f \in l^1(S).$$

Hence

$$\sum_{s \in S} f(s)\varphi(s) = \int_{\hat{S}} \Big(\sum_{s \in S} f(s)\gamma(s)\Big)\, d\mu(\gamma)$$
$$= \sum_{s \in S} f(s) \int_{\hat{S}} \gamma(s)\, d\mu(\gamma) \quad \text{for every} \quad f \in l^1(S)$$

and therefore we conclude that

$$\varphi(s) = \int_{\hat{S}} \gamma(s)\, d\mu(\gamma) \quad \text{for every} \quad s \in S.$$

The uniqueness of μ is a consequence of the Stone-Weierstrass Theorem: the algebra $\widehat{l^1(S)}$ is dense in the algebra of all real continuous functions on $\hat{S}$.

Application 4 (The spectral theorem for normal operators). Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the Banach algebra of all bounded linear operators on $\mathscr{H}$. Consider a subalgebra $A \subset \mathscr{L}(\mathscr{H})$ with the following properties:

(i) A is commutative;

(ii) A is closed;

(iii) If $T \in A$ then $T^* \in A$;

(iv) The identity operator belongs to A.

Let Δ denote the maximal ideal space of A. Since each $T \in A$ is normal it follows that $\| T \| = \| \hat{T} \|_\infty$ for every $T \in A$.

For each pair of vectors $\xi, \eta \in \mathscr{H}$ define a mapping $L_{\xi,\eta} : A \to \mathbf{C}$ by

$$L_{\xi,\eta}(T) = (T\xi, \eta)$$

then we have

$$| L_{\xi,\eta}(T) | \leqslant \| T \| \cdot \| \xi \| \; \| \eta \| = \| \xi \| \; \| \eta \| \cdot \| \hat{T} \|_\infty$$

Therefore by Theorem 1 there exists a bounded complex Radon measure $\mu_{\xi,\eta}$ on Δ such that $\| \mu_{\xi,\eta} \| \leqslant \| \xi \| \cdot \| \mu \|$ and

$$L_{\xi,\eta}(T) = \int_\Delta \hat{T} \, d\mu_{\xi,\eta} \quad \text{for every} \quad T \in A .$$

An application of the Gelfand-Neumark theorem establishes the uniqueness of the measure. The usual construction of a unique resolution of the identity on the Borel sets of Δ can be made based on this formula. A specialization of this formula to a single normal operator leads to the classical spectral theorem. We shall not give the details here since many excellent accounts exist (c.f. Berberian [1], [2], Segal-Kunze [18]). An especially lucid presentation is given in Rudin [16].

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