2. Van der Waerden's Theorem and related topics

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Thus, we hope the reader will forgive us if some (not many, we hope) of the problems turn out to be disappointingly simple.

The chapter titles in the "Monographie" will be:

I. Van der Waerden's theorem and related topics;
II. Covering congruences;
III. Unit fractions;
IV. Bases and related topics;
V. Completeness of sequences and related topics;
VI. Irrationality and transcendence;
VII. Diophantine problems;
VIII. Miscellaneous problems;
IX. Remarks on an earlier paper.

The "earlier paper" referred to in IX is the problem collection of Erdös, "Quelques problèmes de la théorie des nombres", *Monographie de l'Enseignement Math. No. 6* (1963), 81-135. In IX, we give the current status (to the best of our knowledge) on all the problems which appeared there.

2. Van der Waerden's Theorem and related topics

Denote by $W(n)$ the smallest integer such that if the (positive) integers not exceeding $W(n)$ are partitioned arbitrarily into two classes, at least one class always contains an arithmetic progression (A.P.) of length n. The celebrated theorem of van der Waerden [Wa (27)], [Wa (71)], [Gr-Ro (74)] shows that $W(n)$ exists for all n but all known proofs yield upper bounds on $W(n)$ which are extremely weak, e.g., they are not even primitive recursive functions of n. In the other direction, the best lower bound currently available (due to Berlekamp [Ber (68)]) is

$$W(n+1) > n \cdot 2^n$$

for n prime. It would be very desirable to know the truth here. The only values of $W(n)$ known (see [Chv (69)], [St-Sh (78)]) at present are:

$$W(2) = 3\,, \quad W(3) = 9\,, \quad W(4) = 35\,, \quad W(5) = 178\,.$$

Recent results of Paris and Harrington [Par-Har (77)] show that certain combinatorial problems with a somewhat similar flavor (in particular, being variations of Ramsey's Theorem [Ramsey (30)], [Gr-Ro (71)]) do in fact have lower bounds which grow faster than any function which is provably recursive in first-order Peano arithmetic.

More than 40 years ago, Erdös and Turán [Er-Tu (36)], for the purposes of improving the estimates for $W(n)$, introduced the quantity $r_k(n)$, defined to be the least integer r so that if $1 \leqslant a_1 < \ldots < a_r \leqslant n$, then the sequence of a_i's must contain a k-term A.P. The best current bounds [Beh (46)], [Roth (53)], [Mo (53)] on $r_3(n)$ are

$$\frac{n}{\exp\,(c_1 \sqrt{\log\, n})} < r_3(n) < \frac{c_2 n}{\log\, \log\, n}$$

where $c, c_1, c_2, \ldots$ will always denote suitable positive constants. Rankin [Ran (60)] has slightly better bounds for $r_k(n)$, $k > 3$. However, a recent stunning achievement of Szemerédi [Sz (75)] is the proof of the upper bound

$$r_k(n) = o(n)\,.$$

His proof, which uses van der Waerden's theorem, does not give any usable bounds for $W(n)$. This result has also been proved in a rather different way by Furstenberg [Fu (77)] using ergodic theory. This proof also furnishes no estimate for $r_k(n)$. A much shorter version has recently been given by Katznelson and Ornstein (see [Tho (78)]). Perhaps

$$r_k(n) \overset{?}{=} o\left(\frac{n}{(\log\, n)^t}\right)$$

for every t. This would imply as a corollary that for every k there are k primes which form an A.P. The longest A.P. of primes presently known [Weint (77)] has length 17. It is $3430751869 + 87297210t$, $0 \leqslant t \leqslant 16$.

We next mention several conjectures which seem quite deep. They each would imply Szemerédi's theorem, for example.

The first one [1]) is this: Is it true that if a set A of positive integers satisfies $\sum_{a \in A} \frac{1}{a} = \infty$ then A must contain arbitrarily long A.P.'s?

Set

$$\alpha_k = \sup_{A_k} \sum_{a \in A_k} \frac{1}{a}$$

where A_k ranges over all sets of positive integers which do not contain

[1]) One of the authors (P.E.) currently offers US \$3000 for the resolution of this problem.

a k-term A.P. As far as we know, $\alpha_k = \infty$ is possible, but this seems unlikely. The best lower bound known for α_k is due to Gerver [Ge (77)]:

$$\alpha_k \geqslant (1 + o(1))\, k \log k .$$

Trivially,

$$\alpha_k \geqslant \frac{1}{2} \log W(k) .$$

It would be interesting to show that

$$\alpha_k / \log W(k) > \frac{1}{2} + c$$

or even

$$\lim_{k \to \infty} \alpha_k / \log W(k) \to \infty$$

but at present we have no idea how to attack these questions.

The second conjecture is based on the following ideas. For a finite set $X = \{ x_1, ..., x_t \}$, let X^N denote the set of N-tuples $\{ (y_1, ..., y_N) : y_i \in X, 1 \leqslant i \leqslant N \}$. Call a set $P = \{ \bar{p}_1, \bar{p}_2, ..., \bar{p}_t \}$ of t N-tuples $\bar{p}_i \in X^N$ a *line* if the $\bar{p}_i$ have the following property: For each j, $1 \leqslant j \leqslant N$, *either* the jth component of $\bar{p}_i$ is x_i, $1 \leqslant i \leqslant t$, *or* all the jth components of the $\bar{p}_i$ are equal. Since $| P | = t$ then at least one j must satisfy the first condition. It is a theorem of Hales and Jewett [Hale-Je (63)] that for any r, if $N \geqslant N(t, r)$ then for any partition of X^N into r classes, some class must contain a line. This immediately implies van der Waerden's theorem by taking $x_i = i - 1$, $1 \leqslant i \leqslant t$, and letting the N-tuple $(y_1, ..., y_N)$ correspond to the base t expansion of the integer $\sum_{i=1}^{N} y_i t^{i-1}$. In fact, it also implies the higher-dimensional generalizations of van der Waerden's theorem we shall mention shortly. The question now is this: Does the corresponding "density" result hold? In other words, is it true that for each $\varepsilon > 0$ and each integer t, there is an $N(t, \varepsilon)$ so that if $N \geqslant N(t, \varepsilon)$ and R is any subset of X^N satisfying $| R | > \varepsilon t^N$ then R contains a line P? (See also [Mo (70)], [Chv (72)]). For $t = 2$ each line P can be naturally associated with a pair of subsets $A, B \subseteq X$ with $A \subset B$. The truth of the conjecture for $t = 2$ then follows from the theorem of Sperner [Sper (28)] on the maximum size of a family of incomparable subsets of an N-set, namely, that such a family can have at most $\left(\begin{array}{c} N \\ \left[\frac{N}{2} \right] \end{array} \right) = o(2^N)$ sets. However for $t \geqslant 3$ the question is still wide open. Some recent partial results have been given by Brown

[Bro (75)]. It is not even known whether for every c, $\frac{ck^N}{\sqrt{N}}$ points can be chosen without containing a line.

It is natural to ask whether analogues of van der Waerden's theorem hold in higher dimensions, i.e., for any finite subset S of the lattice points of $\mathbf{E}^n$ and any partition of the lattice points of $\mathbf{E}^n$ into two classes, at least one class contains a subset similar to S. That this is indeed the case was first shown by Gallai (see [Rad (33) b]) and independently by Witt [Wit (52)] and by Garsia [Gar (∞)]. The corresponding "density" results, i.e., the analogues of Szemerédi's theorem in higher dimensions, have very recently been proved by Furstenberg and Katznelson [Fu-Ka (78)] using techniques from ergodic theory. These would also follow from the truth of the "line" conjecture previously mentioned. It was previously shown by Szemerédi [Sz (∞)] (using $r_k(n) = o(n)$) that if R is a subset of $\{(i,j): 1 \leqslant i,j \leqslant n\}$ with $|R| \geqslant \varepsilon n^2$ and $n \geqslant n(\varepsilon)$ then R must contain 4 points which form a square. Prior to that, Ajtai and Szemerédi [Aj-Sz (74)] had proved the analogous weaker result for the isoceles right triangle.

A recently very active area deals with various generalizations of the old result of Schur [Schur (16)]: For any partition of $\{1, 2, ..., [r!\,e]\}$ into r classes, the equation $x + y = z$ has a solution entirely in one class. This was generalized (independently) by Rado [Rad (70)], Sanders [San (68)], and Folkman (see [Gr-Ro (71)]) who showed that for any partition of $\mathbf{N}$ into finitely many classes, some class C must contain arbitrarily large sets $\{x_1, x_2, ..., x_k\}$ such that all sums $\sum_{i=1}^{k} \varepsilon_i x_i$, $\varepsilon_i = 0$ or 1, belong to C. However, these results were subsumed by a fundamental result of Hindman [Hi (74)] who showed that under the same hypothesis, some class C must contain an *infinite* set $\{x_1, x_2, ...\}$ such that all finite sums $\sum_{i=1}^{\infty} \varepsilon_i x_i$, $\varepsilon_i = 0$ or 1, belong to C (answering a conjecture of Graham and Rothschild and Sanders). Subsequently, simpler proofs were given by Baumgartner [Bau (74)] and Glazer [Gl (xx)]. Of course, the analogous result also holds for products (by restricting our attention to numbers of the form 2^x). A natural question to ask (see [Er (76) c]) is whether some C must simultaneously contain infinite sets A and B such that all finite sums from A and all finite products from B are in C? Even more, is it possible that we could take $A = B$? In [Hi (xx) a], [Hi (xx) b] Hindman shows that the answer to the first question is yes and the answer to the second question is no. In fact, he constructs a partition of N into two classes such that no infinite set $\{ x_1, x_2, \ldots \}$ has all its finite products and *pair* sums $x_i + x_j$, $i \neq j$, in one class. He also constructs a partition of $\mathbf{N}$ into *seven* classes so that no infinite set $\{ x_1, x_2, \ldots \}$ has all its pair products $x_i x_j$ and pair sums $x_i + x_j$, $i \neq j$, belonging to a single class. Whether arbitrarily large *finite* sets $\{ x_1, \ldots, x_k \}$ with this property can always be found for any partition of $\mathbf{N}$ into finitely many classes is completely open. For a complete and readable account of these and related developments, the reader should consult the survey of Hindman [Hi (79)].

There is a rapidly growing body of results which has appeared recently and which goes under the name of Euclidean Ramsey Theory. The basic question it attacks is this: Given n and r, which configurations $C \subseteq \mathbf{E}^n$ have the property that for any partition of $\mathbf{E}^n$ into r classes, some class must contain a set *isometric* to C. For example, if C consists of 3 points forming a right triangle then a result of Shader [Shad (76)] shows that any partition of $\mathbf{E}^2$ into 2 classes always has a copy of C in at least one of the classes. A similar result also holds for 30° triangles and 150° triangles [Er + 5 (75)]. Note that this is not true if C is a unit equilateral triangle—in this case we simply partition the plane into alternating half open strips of width $\sqrt{3}/2$. The strongest conjecture dealing with this case is that for any partition of $\mathbf{E}^2$ into 2 classes, some class contains congruent copies of *all* 3-point sets with the possible exception of a single equilateral triangle.

A configuration $C \subseteq \mathbf{E}^n$ is called *Ramsey* if for all r, there is an $N(C, r)$ so that for any partition of $\mathbf{E}^N$ with $N \geqslant N(C, r)$ into r classes, some class always contains a subset congruent to C. There are two natural classes which are known to bound the Ramsey configurations. On one hand, it is known [Er + 5 (73)] that the set of the 2^n vertices of any rectangular parallelepided (= brick) is Ramsey (and consequently, so is every subset of a brick, e.g., every acute triangle). On the other hand, it is known [Er + 5 (73)] that every Ramsey configuration must lie on the surface of some sphere S^n. Thus, any set of 3 points in a straight line is not Ramsey (there are partitions of $\mathbf{E}^n$ into 16 classes which avoid having any particular 3 point linear set in one class). Thus, the Ramsey configurations lie between bricks and spherical sets. The unofficial consensus is that they are probably just the (subsets of) bricks but there is no strong evidence for this. Interesting special cases to attack here would be to decide if the vertices of an isosceles 120° triangle or the vertices of a regular pentagon are Ramsey.

Another result of this type more closely related to A.P.'s is the following. It has been shown that there is a large M so that it is possible to partition

$\mathbf{E}^2$ into two sets A and B so that A contains no pair of points with distance 1 and B contains no A.P. of length M. How small can M be made? The only estimate currently known is that $M \leqslant 10000000$ (more or less). In the other direction, it has just been shown by R. Juhász [Ju (79)] that we must have $M \geqslant 5$. In fact, she shows that B must contain a congruent copy of any 4-point set. As a final Euclidean Ramsey question, we mention the following. It was very recently shown by Graham [Gr (xx)] (in response to a question of R. Gurevich [Bab (76)]) that for any r, there is a (very large) number $G(r)$ so that for any partition of the lattice points of the plane into r classes, some class contains the vertices of a right triangle with area *exactly* $G(r)$. It follows from this (see [Gr-Sp (78)] that for any partition of all the points of $\mathbf{E}^2$ into finitely many classes, some class contains the vertices of triangles of each area. The question is: Is this also true for rectangles? or perhaps parallelograms? On the other hand, it is certainly *not* true for rhombuses.

An interesting variation of van der Waerden's theorem is to require that the desired A.P. only hit one class more than the other class by some fixed amount (rather than be completely contained in one class). More precisely, let $f(n, k)$ denote the least integer so that if we divide the integers not exceeding $f(n, k)$ into two classes, there must be an A.P. of length n, say $a + ud$, $0 \leqslant u \leqslant n-1$, with $a + (n-1)d \leqslant f(n, k)$ such that

$$\sum_{u=0}^{n-1} g(a+ud) > k$$

where $g(m)$ is $+1$ if m is in the first class and -1 if m is in the second class. $f(2n, 0)$ has been determined by Spencer [Spen (73)] but we do not have a decent bound for even $f(n, 1)$. It seems likely that $\lim_n W(n)^{1/n} = \infty$ but perhaps $\lim_n f(n, cn)^{1/n} < \infty$. Unfortunately, we cannot even prove $\lim_n f(n, 1)^{1/n} < \infty$. Perhaps this will not be hard but we certainly do not see how to prove $\lim_n f(n, \sqrt{n})^{1/n} < \infty$. Define

$$F(x) = \min_g \max \left| \sum g(a+kd) \right|$$

where the maximum is taken over all A.P.'s whose terms are positive integers and the minimum is taken over all functions $g: \mathbf{Z} \to \{-1, 1\}$. Roth [Roth (64)] proved that

$$F(x) > cx^{1/4}$$

and conjectured that for every $\varepsilon > 0$, $F(x) > x^{1/2-\varepsilon}$ for $x > x_0(\varepsilon)$. In the other direction Spencer [Spen (72)] showed that

$$F(x) < cx^{1/2} \frac{\log \log x}{\log x}.$$

However, Sarkozy (see [Er-Sp (74)]) subsequently showed that

$$F(x) = O\left((x \log x)^{1/3}\right),$$

disproving the conjecture of Roth.

Cantor, Erdös, Schreiber and Straus [Er (66)] (also see [Er (73) b]) proved that there is a $g(n) = \pm 1$ for which

$$\max_{\substack{a, m \\ 1 \leq b \leq d}} \left| \sum_{k=1}^{m} g(a+kb) \right| < h(d)$$

for a certain function $h(d)$. They showed that $h(d) < cd!$ No good lower bound for $h(d)$ is known. As far as we know the following related more general problem is still open. Let $A_k = \{ a_1^{(k)} < a_2^{(k)} < \dots \}$, $k = 1, 2, \dots$ be an infinite class of infinite sets of integers. Does there exist a function $F(d)$ (depending on the sequences A_k) so that for a suitable $g(n) = \pm 1$

$$\max_{m, 1 \leq k \leq d} \left| \sum_{i=1}^{m} g(a_i^{(k)}) \right| < F(d) \ ?$$

It seems certain that the answer is affirmative.

Finally, is it true that for every c, there exist d and m so that

$$\left| \sum_{k=1}^{m} g(kd) \right| > c \ ?$$

The best we could hope for here is that

$$\max_{md \leq n} \left| \sum_{k=1}^{m} g(kd) \right| > c \log n .$$

We remark that these questions can also be asked for functions $g(n)$ which take kth roots of unity as values rather than just ± 1. However, very little is yet known for this case.

Another interesting problem: For $r < s$ denote by $f_r(n; s)$ the smallest integer so that every sequence of integers of n terms which contains $f_r(n; s)$ A.P.'s of length r must also contain an A.P. of length s. Perhaps for s

$= o(\log n)$, $f_3(n; s) = o(n^2)$; this is certainly false for $s > \varepsilon \log n$. At present we cannot even prove $f_3(n; 4) = o(n^2)$.

Abbott, Liu and Riddell [Ab-Li-Ri (74)] define $g_k(n)$ as the largest integer so that among any n real numbers one can always find $g_k(n)$ of them which do not contain an A.P. of length k. It is certainly possible to have $g_k(n) < r_k(n)$; in fact, Riddell shows that $g_3(14) = 7$ but $r_3(14) = 8$. It is not known if $g_3(n) < r_3(n)$ for infinitely many n. It follows from a very interesting general theorem of Komlós, Sulyok and Szemerédi [Kom-Su-Sz (75)] that $g_3(n) > cr_3(n)$. Perhaps $\lim_{n\to\infty} \frac{r_3(n)}{g_3(n)} = 1$. Szemerédi points out that it is not even known if $\frac{r_4(n)}{r_3(n)} \to \infty$.

The following question is due to F. Cohen. Determine or estimate a function $h(d)$ so that if we split the integers into two classes, at least one class contains for infinitely many d an A.P. of difference d and length at least $h(d)$. Erdös observed that $h(d) < cd$ is forced and Petruska and Szemerédi [Pe-Sz (∞)] strengthened this by showing that $h(d) < cd^{1/2}$. Very recently, J. Beck [Bec (xx)] showed $h(d) < \frac{(1+o(1)) \log d}{\log 2}$. The theorem of van der Waerden shows that $h(d) \to \infty$ with d but we currently have no usable lower bound for $h(d)$.

Define $H(n)$ to the smallest integer so that for any partition of the integers $\{1, 2, \dots, H(n)\}$ into any number of disjoint classes, there is always an n-term arithmetic progression all of whose terms either belong to one class or all different classes. The existence of $H(n)$ is guaranteed by Szemerédi's theorem. In fact it is easy to show $H(n)^{1/n} \to \infty$; to show $H(n)^{1/n}/n \to \infty$ might be much harder. What can be said about small values of $H(n)$?

Is it true that for any partition of the pairs of positive integers into two classes, the sums $\sum_{x \in X} \frac{1}{\log x}$ are unbounded where X ranges over all subsets which have all pairs belonging to one class?

It was conjectured by Erdös that for every $\varepsilon > 0$ there is a t_ε so that the number of squares in any A.P. $a + kd$, $0 \leqslant k \leqslant t_\varepsilon$, is less than $\varepsilon t_\varepsilon$. This follows from Szemerédi's result $r_k(n) = o(n)$; in fact, his earlier result $r_4(n) = o(n)$ (see [Sz (69)]) suffices for this purpose. Rudin conjectured that there is an absolute constant c so that the number of squares in $a + kd$, $0 \leqslant k \leqslant t$, is less than $c\sqrt{t}$. Rudin's conjecture is still open.

Denote by $F(n)$ the largest integer r for which there is a non-averaging sequence $1 \leqslant a_1 < \ldots < a_r \leqslant n$, i.e., no a_i is the arithmetic mean of other a_j's. Erdös and Straus [Er-Str (70)] proved

$$\exp(c\sqrt{\log n}) < F(n) < n^{2/3}.$$

However, Abbott [Ab (75)] just proved the unexpected result

$$F(n) > n^{1/10}.$$

It would be nice to know what the correct exponent is here.

It seems to be difficult to state reasonable conditions which imply the existence of an infinite A.P. in a set of integers. For example, because there are only countably many infinite A.P.'s, then for any sequence a_n, there is a sequence b_n with $b_n > a_n$ so that the b_n's hit every infinite A.P. It is not difficult to show that for any sequence $B = (b_1, b_2, \ldots)$ with $b_1 \geqslant 5$ and $b_{i+1} \geqslant 2b_i$ there is a set $A = \{a_1, a_2, \ldots\}$ with $2 \leqslant a_{k+1} - a_k \leqslant 3$ for all k so that for all i, $b_i \notin A + A = \{a + a' : a, a' \in A\}$. Whether such behavior can hold for $A + A + A$ (or more summands) is not known.

The situation is completely different, however, when one considers *generalized* A.P.'s

$$S(\alpha, \beta) = (a_1, a_2, \ldots, a_n, \ldots).$$

A generalized A.P. is formed by $a_n = [\alpha n + \beta]$ for given real $\alpha \neq 0$ and β. It follows from results of Graham and Sós [Gr-Só (xx)] that if $b_{n+1}/b_n \geqslant c > 2$ then the complement of the b_k's contains an infinite generalized A.P. This has very recently been strengthened by Pollington [Poll (xx)] who proved that there is no sequence b_n hitting every generalized A.P. with $b_{k+1}/b_k \geqslant c > 1$ for all k. On the other hand, for any sequence c_n there exist sequences b_n with $b_n > c_n$ which hit every generalized A.P.

Of course, almost any question dealing with A.P.'s can also be asked about generalized A.P.'s. For example, can we get better (much better?) bounds for van der Waerden's theorem when we allow generalized A.P.'s? This question has not yet been investigated so far.

The generalized A.P.'s $S(\alpha) = S(\alpha, 0) = \{[\alpha n] : n = 1, 2, \ldots\}$ have an extensive literature (e.g., see [Frae (69)], [Ni (63)], [Frae-Le-Sh (72)], [Gr-Li-Li (78)] and especially [Stol (76)]). One of the earliest results [Bea (26)] asserts that $S(\alpha_1)$ and $S(\alpha_2)$ disjointly cover the positive integers iff the α_i are irrational and $\dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2} = 1$. An old result of Uspensky [U (27)], [Gr (63)] shows that $\mathbf{Z}^+$ can never be partitioned into three or more disjoint $S(\alpha_i)$; in fact, for any three $S(\alpha_i)$, some pair of them must have infinitely many common elements.

This situation does not hold for general $S(\alpha, \beta)$ however. For example, $S(2;0)$, $S(4;1)$, $S(4;3)$ and $S\left(\frac{7}{4};0\right)$, $S\left(\frac{7}{2};\frac{5}{2}\right)$, $S(7;4)$ both form decompositions of the nonnegative integers. Of course, more generally, if

$$\mathbf{Z} = \sum_{i=1}^{m} S(a_i; b_i), \; a_i, \; b_i \in \mathbf{Z}$$

is a decomposition of $\mathbf{Z}$ into disjoint A.P.'s of integers then $S(\alpha; \beta) = \sum_{i=1}^{m} S(a_i\alpha; \beta + \alpha b_i)$. It has been shown by Graham [Gr (73)] that if $m \geqslant 3$, $\mathbf{Z}^+ = \sum_{i=1}^{m} S(\alpha_i; \beta_i)$ and some α_i is irrational then the $S(\alpha_i; \beta_i)$ must be generated from two disjoint $S(\gamma_i; 0)$ which cover $\mathbf{Z}^+$ by transformations of this type. In particular, it follows from the theorem of Mirsky and Newman (see [Er (50)]) that for some $i \neq j$, $\alpha_i = \alpha_j$. Curiously, the situation is much less well understood when all the α_i are rational. A striking conjecture of Fraenkel [Frae (73)] asserts that for any such decomposition (with $m \geqslant 3$) with all α_i distinct, we must have $\{\alpha_1, \ldots, \alpha_m\} = \left\{\frac{2^m - 1}{2^k} : 0 \leqslant k < m\right\}$.

One can ask how sparse (in some sense) a set S of integers can be and still have the property that for any decomposition of S into r classes, some class must contain an A.P. of length k. Of course, since the multiples of any d have this property, we must be more precise about what we mean by sparse. For example, we might ask whether such an S exists which itself contains no A.P. of length $k + 1$. That such S's exist was first shown by Spencer [Spen (75)] (using the previously mentioned theorem of Hales and Jewett) and independently by Nešetril and Rödl [Neš-Röd (∞)].

An old theorem of Brauer [Bra (28)] (also see [Ab-Ha (72)], [Rad (33) a]) proves a stronger form of van der Waerden's theorem in which not only must one of the classes contain an A.P. of length r, say, $a + kd$, $0 \leqslant k < r$, but also the common difference d as well. However, the analogue of Szemerédi's theorem does not hold for this case—we can find sets of positive density which do not contain a k-term A.P. together with its difference. For example, the set of odd integers cannot contain a, $a + d$ and d. However, the densest subset R of $\{1, 2, \ldots, n\}$ not containing a k-term A.P. and its difference has recently been determined by Graham, Spencer and Witsenhausen [Gr-Sp-Wi (77)]. Their result shows that any such R must satisfy $|R| \leqslant n - \left[\frac{n}{k}\right]$ (and this is best possible). Almost all cases of this type of problem remain open. One of the simplest is this: Let R_n be a maximum subset of $\{1, 2, ..., n\}$ with the property that for no x are x, $2x$ and $3x$ all in R_n. What is $\lambda = \lim_n \frac{|R_n|}{n}$? (Its existence is known). In particular, prove that λ is irrational. Of course, one could ask these questions for infinite sets of integers. For example, if $a_1 < a_2 < ...$ is an infinite sequence of integers such that for no x are x, $2x$, $3x$ all a_i's, then how large can the density of the a's be (if it exists)? Can the upper density be larger?

In a different direction, one could ask how many subsets of $\{1, 2, ..., n\}$, say $S_1, ..., S_t$, can one have so that for all $i \neq j$, $S_i \cap S_j$ is an A.P. Simonovits, Sós and Graham [Gr-Si-Só (xx)] have recently shown that $t \leqslant \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + 1$ and this is best possible. If $S_i \cap S_j$ must be a *nonempty* A.P. then Simonovits and Sós [Sim-Sós (xx)] have given an ingenious proof that $t < cn^2$. It is conjectured in this case that the maximum families form strong Δ-systems, i.e., the S_i are just all the finite A.P.'s in $\{1, 2, ..., n\}$ containing a particular element, presumably the integer $\left[\frac{n}{2}\right]$.

An easy consequence of van der Waerden's theorem is the following: If $A = (a_1, a_2, ...)$ is an increasing infinite sequence of integers with $a_{k+1} - a_k$ bounded then A contains arbitrarily long A.P.'s (see [Kak-Mo (30)]). The analogous questions in higher dimensions are not yet completely settled. For example, let $p_i = (x_i, y_i)$, $i = 1, 2, ...$ be an infinite set of lattice points in $\mathbf{E}^2$ so that $p_{i+1} - p_i$ is either (0, 1) or (1, 0). Must the p_i contain arbitrarily long A.P.'s? Surprisingly, the answer is no. It is possible to use the strongly non-repetitive sequences of Dekking [Dek (79)] (also, see Pleasants [Ple (70)], [Bro (71)]) to construct such a sequence of p_i having no 5-term A.P. On the other hand, it is not hard to see that 4-term A.P.'s cannot be avoided. Similar techniques can be used to show that there are increasing unit-step sequences of lattice points in $\mathbf{E}^5$ containing no 3-term A.P. Whether or not this can be done in $\mathbf{E}^3$ or $\mathbf{E}^4$ is not known. Pomerance [Pom (xx)] has recently shown that if the *average* step size is bounded, there must be arbitrarily large sets of the a_k which lie on some line. In fact, he shows somewhat more, e.g., that the same conclusion holds for the points

(n, p_n) where p_n denotes the n^{th} prime [Pom (79)] (however, the proof of the former does not provide effective bounds).

Gerver and Ramsey [Ge-Ra (xx)] give an effective estimate for the following special case. Suppose $S \subseteq \mathbf{Z}^2$ is finite and let $A = (a_1, a_2, \ldots, a_N)$ be a sequence of lattice points with $a_{k+1} - a_k \in S$ for all $k < N$. (Such a sequence is called an *S-walk*). Then for any $\varepsilon > 0$, if $N \geqslant N_0(M, \varepsilon)$ where M denotes the maximum distance of any point in S from the origin, A must contain at least $C(M, \varepsilon) (\log N)^{1/4-\varepsilon}$ collinear points. On the other hand, such a result does not hold for $\mathbf{Z}^3$. In particular, they construct an infinite sequence $B = (b_1, b_2, \ldots)$ of lattice points in $\mathbf{Z}^3$ for which $b_{k+1} - b_k$ is a unit vector for all k and such that B has at most 5^{11} collinear points. They conjecture that 3 is actually the correct bound for their construction. It is not known whether there is an infinite S-walk in $\mathbf{Z}^3$ for S finite which has no three points collinear.

A number of interesting questions involving A.P.'s come up in the following way. Let us say that a (possibly infinite) sequence $(a_1, a_2, \ldots)$ has a *monotone* A.P. of length k if for some choice of indices $i_1 < i_2 < \ldots < i_k$, the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is either an increasing or a decreasing A.P. It has often been noted that it is possible to arrange any finite set of integers into a sequence containing no monotone A.P. of length 3. Essentially, this can be done by placing all the odd elements to the left of all the even elements, arranging (by induction) the odds and the evens individually to have no monotone A.P.'s of length 3 and using the fact that the first and last terms of a 3-term A.P. must have the same parity. If $M(n)$ denotes the number of permutations of $\{1, 2, \ldots, n\}$ having no monotone 3-term A.P., it has been shown by Davis, Entringer, Graham and Simmons [Dav+3 (77)] that

$$M(n) \geqslant 2^{n-1}, \quad M(2n-1) \leqslant (n!)^2, \quad M(2n+1) \leqslant (n+1)\,(n!)^2$$

It would be interesting to know if $M(n)^{1/n}$ is bounded or even tends to a limit. The situation for permutations of infinite sets is different. It has been shown by the above mentioned authors that any permutation of $\mathbf{Z}^+$ contains an increasing 3-term A.P. but that there exist permutations of $\mathbf{Z}^+$ which have no monotone 5-term A.P.'s. The question of whether or not monotone 4-term A.P.'s must occur is currently completely open. If one is allowed to arrange $\mathbf{Z}^+$ into a doubly-infinite sequence $\ldots, a_{-1}, a_0, a_1, \ldots$ then monotone 3-term A.P.'s must still occur but it is now possible to prevent those of length 4. If the elements to be permuted are *all* the integers rather than just the positive integers, less is known. It is known [Odd (75)] that monotone 7-term A.P.'s can be stopped in the singly-infinite case. We should note that the modular analogues of these problems have been studied by Nathanson [Na (77)].

Must any ordering of the reals contain a monotone k-term arithmetic progression for every k?

We conclude this topic with a very annoying question: Is it possible to partition $\mathbf{Z}^+$ into two sets, each of which can be permuted to avoid monotone 3-term A.P.'s? If we are allowed three sets, this is possible; the corresponding situation for $\mathbf{Z}$ has not been investigated.

It is not difficult to find finite sets $A = \{ a_1, ..., a_n \}$ with the property that for any two elements $a_i, a_j \in A$, there is an $a_k \in A$ so that $\{ a_i, a_j, a_k \}$ forms an A.P., e.g., $\{ 1, 2, 3 \}$ and $\{ 1, 3, 4, 5, 7 \}$. In fact, it is not difficult to show that up to some affine transformation $x \to ax + b$, these are the only such sets. It follows from this that the analogue of Sylvester's theorem holds for A.P.'s, i.e., no finite set A has the property that every 3 terms of A belong to some A.P. in A. Suppose one only requires that for every choice of k terms from A, some 3 (or m) of them belongs to an A.P. in A. Can those A now be characterized? One might also ask these questions for generalized A.P.'s as well where we would expect much richer classes of A's because of the greater number of generalized A.P.'s.

Stanley has raised the following question (generalizing an earlier question of Szekeres (see [Er-Tu (36)])). Starting with $a_0 = 0$, $a_1 = a$, form the infinite sequence $a_0, a_1, a_2, a_3, ...$ recursively by choosing a_{n+1} to be the least integer exceeding a_n which can be adjoined so that no 3-term A.P. is formed. Can the a_k be explicitly determined? For example, if $a = 1$ then the a_k are just those integers which have no 2 in their base 3 expansion. Similar characterizations are known when $a = 3^r$ and $a = 2 \cdot 3^r$ (see [Odl-Sta (78)]). For these cases, if $\alpha = \dfrac{\log 3}{\log 2}$ then $\liminf\limits_{n} \dfrac{a_n}{n^\alpha} = 1/2$, $\limsup\limits_{n} \dfrac{a_n}{n^\alpha} = 1$. However, the case of $a = 4$ (and all other values not equal to 3^r or $2 \cdot 3^r$) seems to be of a completely different character. There are currently no conjectures for the a_k in this case.

Hoffman, Klarner and Rado [Kl-Ra (73)], [Kl-Ra (74)], [Hoff-Kl (xx)], [Hoff (76)] have obtained many interesting results on the following problem: Let R denote a set of linear operations on the set of nonnegative integers, each of the type $\rho(x_1, ..., x_r) = m_0 + m_1x_1 + ... + m_rx_r$. Given a set A of positive integers, let $< R : A >$ denote the smallest set containing A which is closed under all operations in R. What is the structure of $< R : A >$? Two basic results here are:

(i) For any infinite set B there is a finite set A such that $< R : B > = < R : A >$ whenever at least one $\rho(x_1, ..., x_2) = m_0 + m_1 x_1 + ... + m_r x_r$ has $(m_1, ..., m_r) = 1$.

(ii) If $R = \{ m_0 + m_1 x_1 + ... + m_r x_r \}$ and all m_i are positive then $< R : A >$ is a finite union of infinite A.P.'s, again when $(m_1, ..., m_r) = 1$.

The special case that $R = \{ a_1 x + b_1, ..., a_r x + b_r \}$ is particularly interesting. It has been shown by Erdös, Klarner and Rado (see [Kl-Ra (74)]) that if $\sum_{k=1}^{r} \frac{1}{a_k} < 1$ then $< R : A >$ has density 0. The situation in which $\sum_{k=1}^{r} \frac{1}{a_k} = 1$ is not yet completely understood. This depends, for example, on when the set of transformations $x \to a_i x + b_i$, $1 \leqslant i \leqslant r$, generates a free semigroup under composition. The reader should consult the relevant references for numerous other results and questions.

Erdös asked: If S is a set of real numbers which does not contain a 3-term A.P. then must the complement of S contain an infinite A.P.? R. O. Davies (unpublished) showed that assuming the Continuum Hypothesis the answer is no; Baumgartner [Bau (75)] proved the same thing without assuming the Continuum Hypothesis. Baumgartner also proved the conjecture of Erdös that if A is a sequence of positive integers with all sums $a + a'$ distinct for $a, a' \in A$ then the complement of A contains an infinite A.P. Of course, many generalizations are possible.

Can one prove that the longest arithmetic progression

$$\{ a+kd : 0 \leqslant k \leqslant t \}$$

with $a + kt < x$, which consists entirely of primes satisfies $t = o(\log x)$? Only $t \leqslant (1+o(1)) \log x$ is clear; this follows from the prime number theorem. Suppose that at least ct of the terms are prime. It is not hard to see that $t < (\log x)^{\alpha(c)}$ where $\alpha(c) \to \infty$ as $c \to o$. If there is any justice $\alpha(c)$ should not tend to infinity. If we take $t = \log x$ perhaps the number of primes tends to 0 uniformly in d.