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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **11.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-50373

ON LIE'S HIGHER SPHERE GEOMETRY

by Jay P. FILLMORE

1. INTRODUCTION

In this paper we draw together two theories having their roots in the ideas of S. Lie over a century ago: *Lie's higher sphere geometry*, with its famous *line-sphere transformation* [5], and the theory of Lie groups, especially the description of a geometry by global Lie groups [1]. Indeed, not until the 1960s, with the appearance of W. M. Boothby's description of homogeneous contact manifolds [1, 2] and with the appearance of parabolic subgroups, could this connection be established. One can now say, in terms of Lie groups, that the three-dimensional complex line and sphere geometries are isomorphic and that *the real line and sphere geometries are two distinct real forms of one geometry*. Furthermore, the line-sphere transformation gives explicitly the isomorphism of the complex forms.

In Section 2 we summarize the formulation of Boothby's theory for algebraic homogeneous contact manifolds and make some observations about their real forms. The classical contact manifolds of complex co-directions in projective space and of Lie's higher sphere geometry are described in general in terms of this theory in Sections 3 and 4. Finally, in Section 5, the connection with Plücker's line geometry in three dimensions is established, and the line-sphere transformation is brought into perspective. This introduction continues with an overview of F. Klein's formulation of Lie's theory [5, 6], Boothby's theory, and their connection.

To a line in complex projective space P^3 may be assigned Plücker coordinates

$$\xi_1 = p_{12}, \ \xi_2 = p_{31}, \ \xi_3 = p_{23},$$
$$\xi_4 = p_{03}, \ \xi_5 = p_{02}, \ \xi_6 = p_{01},$$

[6, §20]. These coordinates satisfy

$$\xi_1\xi_4 + \xi_2\xi_5 + \xi_3\xi_6 = 0,$$

[1] This description of Lie's higher sphere geometry in terms of Lie groups answers a question posed in 1965 by S. SASAKI [9, p. 173].

and hence lines in P^3 correspond to points of a quadric Ω^4 in P^5. Two lines in P^3 intersect when their corresponding points on Ω^4 are conjugate. A surface element in P^3, a point and incident plane, becomes the pencil of lines passing through the point and lying in the plane; this corresponds to a line lying in Ω^4. The space of surface elements in P^3 thus corresponds to the space of lines in Ω^4. The projectivities of P^5 which preserve the quadric Ω^4 permute the lines of Ω^4 and hence the surface elements of P^3. Moreover, these projectivities preserve the condition, between two surface elements at infinitesimally adjacent points, that a point of one lies on the plane of the other; hence they are contact transformations of P^3.

To a sphere

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + C = 0$$

in complex Euclidean space E^3, with center at $x = a$, $y = b$, $z = c$ and radius

$$r^2 = a^2 + b^2 + c^2 - C,$$

the sign of r corresponding to an "orientation", may be assigned homogeneous coordinates

$$a = \frac{\alpha}{\nu}, \quad b = \frac{\beta}{\nu}, \quad c = \frac{\gamma}{\nu}, \quad r = \frac{\lambda}{\nu}, \quad C = \frac{\mu}{\nu},$$

[6, §25]. These coordinates satisfy

$$\alpha^2 + \beta^2 + \gamma^2 - \lambda^2 - \mu\nu = 0,$$

and hence oriented spheres in E^3 correspond to certain points of a quadric Ψ^4 in P^5; if spheres which are points or planes or which have centers at infinity are included, all points of Ψ^4 are obtained. Two spheres in E^3 are tangent at a point, orientations taken into account, when their corresponding points on Ψ^4 are conjugate. An "oriented" surface element in E^3, a point and incident oriented plane, becomes the pencil of spheres tangent to the plane at the point; this corresponds to a line lying in Ψ^4. The space of oriented surface elements of E^3 thus corresponds to the space of lines in Ψ^4. The projectivities of P^5 which preserve the quadric Ψ^4 permute the lines of Ψ^4 and hence the oriented surface elements of E^3. Moreover, these projectivities are contact transformations of E^3.

The line-sphere transformation, discovered by Lie, is given by

$$\begin{aligned} \xi_1 &= \alpha + \sqrt{-1}\,\beta, & \xi_4 &= \alpha - \sqrt{-1}\,\beta, \\ \xi_2 &= \gamma + \lambda, & \xi_5 &= \gamma - \lambda, \\ \xi_3 &= \mu, & \xi_6 &= -\nu, \end{aligned}$$

as formulated by Klein [6, §70]. This makes correspond points of the quadric Ω^4 of signature $(+++---)$ and points of the quadric Ψ^4 of signature $(++++--)$. Conjugate points correspond to conjugate points, and a line in one quadric corresponds to a line in the other. Thus, surface elements in P^3 correspond to oriented surface elements in E^3 and this correspondence is a "contact transformation".

Now, classically a *contact transformation* in P^3 or E^3 is a transformation on the 5-dimensional space of surface elements which preserves, up to a non-vanishing multiple, a maximal rank Pfaffian form

$$\omega = dz - pdx - qdy\,,$$

[6, §63], where the coordinates x, y, z, p, q describe the surface element consisting of the plane

$$z' - z = p(x'-x) + q(y'-y)$$

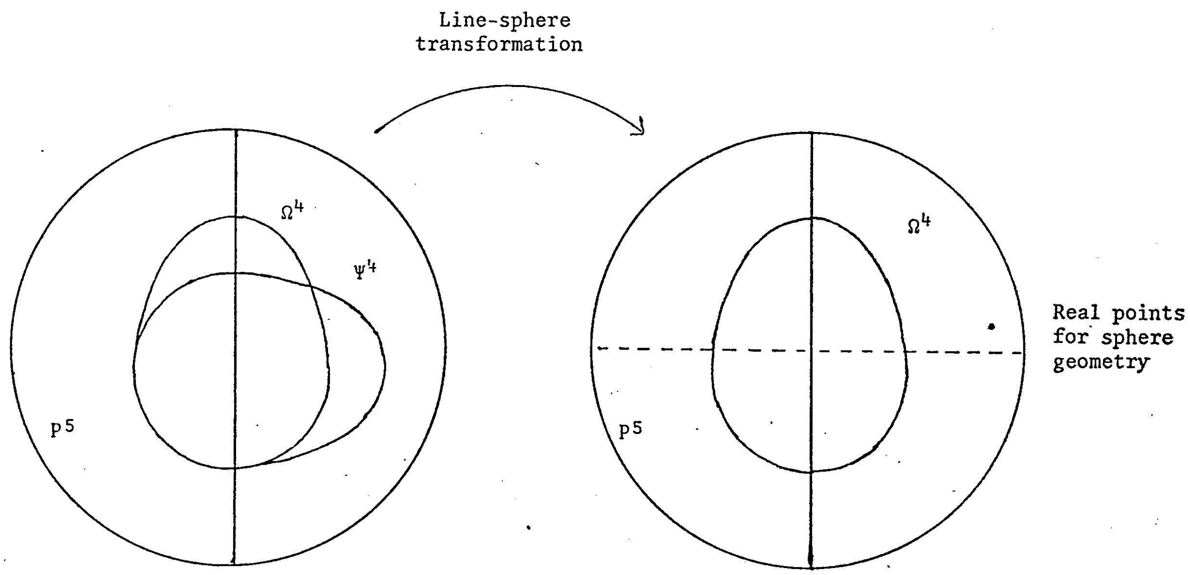
at the point (x, y, z). The condition $\omega = 0$, that at two infinitesimally adjacent points the point of one surface element lies on the plane of the other, is preserved by a contact transformation. The appropriate spaces for the line-sphere transformation are the 5-dimensional spaces of lines in Ω^4 and lines in Ψ^4. Exhibiting the Pfaffian forms and examining the effect of the line-sphere transformation on them may be done systematically by observing that these spaces are *homogeneous.*

Boothby's description of compact homogeneous complex contact manifolds [1, 2; and 7, §2] constructs for each type of simple complex Lie algebra $\mathfrak{g}$: a connected centerless simple Lie groups G having Lie algebra $\mathfrak{g}$, a parabolic subgroup P of G, and a Pfaffian form ω on a principal $\mathbf{C}^*$-bundle over G/P, so that G/P, with ω pulled down by local sections, is a compact complex contact manifold, homogeneous under the identity component G of the group of all its contact automorphisms. Every such contact manifold is so obtained uniquely up to isomorphism. This construction yields, for the classical simple Lie algebras:

A_n	projective cotangent bundle of P^n—the classical space of incident point-hyperplane pairs in P^n,
B_l and D_l	space of lines in a quadric,
C_l	odd-dimensional projective space P^{2l+1},

[1, (7.1)]. The isomorphism $A_3 \simeq D_3$ arises from the description of surface elements in P^3 as lines in Ω^4 by Plücker coordinates. Since the complex quadrics Ω^4 and Ψ^4 both have groups of projectivities of the type D_3, the contact manifolds of line geometry and sphere geometry, when viewed as the spaces of lines in Ω^4 and Ψ^4 respectively, are necessarily *the same*, that is, isomorphic.

When Boothby's description of homogeneous contact manifolds is refined, using J. A. Wolf's theory of complex flag manifolds [8, Ch. I], to include their real forms, line geometry and sphere geometry are *no longer the same*, but, as was classically recognized [6, §25], are obtained from the real forms $PSO\,(3,3;\mathbf{R})$ and $PSO\,(4,2;\mathbf{R})$ of $PSO\,(3,3;\mathbf{C})$ and $PSO\,(4,2;\mathbf{C})$, where the quadratic forms defining these projective special orthogonal groups are those of the quadrics Ω^4 and Ψ^4. Now, $PSO\,(3,3;\mathbf{C})$ and $PSO\,(4,2;\mathbf{C})$ are isomorphic, so the corresponding complex contact manifolds are isomorphic; in fact, these groups, are conjugate in $PSL\,(6;\mathbf{C})$ by the matrix of Klein's description of the line-sphere transformation. Viewed another way, $PSO\,(3,3;\mathbf{R})$ and $PSO\,(4,2;\mathbf{R})$ correspond to two real forms of $PSO\,(3,3;\mathbf{C})$ defined by two complex conjugations. Consequently, *real line geometry and real sphere geometry are two distinct real forms of complex line geometry*. The line-sphere transformation then corresponds to an automorphism of $PSO\,(3,3;\mathbf{C})$ connecting the two complex conjugations.



Real points of P^5.

Real points for line geometry

2. Homogeneous contact manifolds

We formulate the notion of contact manifold in terms of complex analytic manifolds; the definitions apply equally to real smooth manifolds. Especially, if the complex analytic manifold is a smooth algebraic variety defined over **R** and if its various structures are defined over **R**, then the elementary assertions here apply to the set of real points of the variety.

Throughout this section we indicate proofs only when they differ from those of Boothby [1, 2] or Wolf [7, 8].

2.1. A complex contact manifold is a complex manifold M of odd dimension $2n-1$ together with a *complex contact structure* which is prescribed by a family $\{(U_\alpha, \omega_\alpha)\}$ consisting of an open cover $\{U_\alpha\}$ of M and holomorphic Pfaffian forms ω_α on U_α satisfying:

(i) $\omega_\alpha \wedge (d\omega_\alpha)^{n-1}$ does not vanish on U_α, i.e., ω_α is of maximal rank;

(ii) if $U_\alpha \cap U_\beta$ is not empty, then $\omega_\beta = f_{\beta\alpha}\omega_\alpha$ on $U_\alpha \cap U_\beta$ with $f_{\beta\alpha}$ holomorphic and non-vanishing; and

(iii) the family $\{(U_\alpha, \omega_\alpha)\}$ is maximal with respect to (i) and (ii).

A holomorphic map $g\colon M \to M'$ between two contact manifolds is a *contact transformation* if $\{(g^{-1}U'_\alpha, g^* \omega'_\alpha)\}$ is contained in $\{(U_\alpha, \omega_\alpha)\}$. [1, §2].

2.2. Let M be the space of hypersurface elements in complex Euclidean space E^n whose hyperplanes meet the x_n-axis, that is, points $(x_1, \ldots, x_n)$ and incident hyperplanes

$$x'_n - x_n = p_1(x'_1 - x_1) + \ldots + p_{n-1}(x'_{n-1} - x_{n-1}),$$

where primes denote running coordinates. The single set of coordinates $x_1, \ldots, x_n, p_1, \ldots, p_{n-1}$ on M together with the Pfaffian form

$$\omega = dx_n - p_1 dx_1 - \ldots - p_{n-1} dx_{n-1}$$

suffices to define a contact structure on M [6, §63]. This is *the* classical contact manifold to which we will relate all others.

2.3. The contact structure on a complex manifold M has been formulated by S. Kobayashi in terms of a principal $\mathbf{C}^*$-bundle over M [1, §2 and 7, §2]. Let $\{ (U_\alpha, \omega_\alpha) \}$ be a contact structure on M, so that $\omega_\beta = f_{\beta\alpha}\omega_\alpha$ on $U_\alpha \cap U_\beta$. Define the holomorphic principal $\mathbf{C}^*$-bundle $\pi\colon B \to M$ using the transition functions $f_{\beta\alpha}{}^{-1} = \dfrac{1}{f_{\beta\alpha}}$ on $U_\alpha \cap U_\beta$; $\pi^{-1}(U_\alpha)$ is $U_\alpha \times \mathbf{C}^*$ and, with coordinate z_α on $\mathbf{C}^*$, $z_\beta = f_{\beta\alpha}{}^{-1} z_\alpha$ on $U_\alpha \cap U_\beta$. ω_α on U_α pulls back by π^* to a Pfaffian form on $\pi^{-1}(U_\alpha)$, again denoted ω_α. On $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ we have $\omega_\beta = f_{\beta\alpha}\,\omega_\alpha$ and $z_\beta = f_{\beta\alpha}{}^{-1} z_\alpha$ so that $z_\alpha\omega_\alpha = z_\beta\omega_\beta$; the $z_\alpha\omega_\alpha$ hence define a holomorphic Pfaffian form ω on B. Let R_a, a in $\mathbf{C}^*$, denote the right action of $\mathbf{C}^*$ on B. The Pfaffian form ω satisfies:

(a) $(d\omega)^n$ does not vanish on B;

(b) ω vanishes on vectors tangent to the fibers of B; and

(c) $R_a{}^*\omega = a\omega$, a in $\mathbf{C}^*$ [1, (2.1)].

Conversely, a holomorphic principal $\mathbf{C}^*$-bundle B over M together with a holomorphic Pfaffian form ω satisfying (a, b, c) defines a contact structure on M. The ω_α on U_α are obtained by pulling down ω by sections of B over U_α.

Complex contact structures on M correspond uniquely to principal $\mathbf{C}^*$-bundles $\pi\colon B \to M$ equipped with a Pfaffian form ω satisfying (a, b, c), up to isomorphism [1, (2.1)]. Contact transformations $M \to M'$ are exactly those homomorphisms $g\colon B \to B'$, $\pi' \circ g = g \circ \pi$ and $R'_a \circ g = g \circ R_a$, satisfying $g^*\omega' = \omega$. Consequently, contact automorphisms of M correspond to bundle automorphisms g of B which preserve ω: $g^*\omega = \omega$ [1, (3.1)].

In case M is *compact*, its group of all contact automorphisms is a *complex Lie group* which acts holomorphically on B [1, (3.2) and 2, §1].

2.4. Let V be a complex manifold of dimension n and M the bundle of complex co-directions of V, that is, M is obtained from the bundle B which is the cotangent bundle of V less its zero section by passing to the projective space of each fiber. B is a principal $\mathbf{C}^*$-bundle over M. If $x_1, \ldots, x_n$ are coordinates on an open set U of V, ξ in B may be written over U as

$$\xi = u_1(\xi)\,dx_1 + \ldots + u_n(\xi)\,dx_n\,,$$

where the functions $u_i(\xi)$ are homogeneous of degree one; the functions $x_1, \ldots, x_n$, $u_1, \ldots, u_n$ define coordinates in B over U. The Pfaffian form

$$\omega = u_1 dx_1 + \ldots + u_n dx_n$$

on B satisfies (a, b, c) of 2.3 and hence defines a contact structure on M. Again, this is classical [6, §63, p. 242]. We may cover M over U by open sets where some u_i is not zero, say $u_n \neq 0$, and then set

$$u_1 = -p_1, \ldots, u_{n-1} = -p_{n-1}, u_n = 1;$$

this gives a section of B over this open set and ω pulls down to

$$dx_n - p_1 dx_1 - \ldots - p_{n-1} dx_{n-1},$$

the description in 2.2.

2.5. Let V of 2.4 be complex projective space P^n. Points of P^n are described by homogeneous coordinates $x_0, \ldots, x_n$, written as a column vector $x = {}^t(x_1, \ldots, x_n)$, and hyperplanes

$$ux' = u_0 x_0' + \ldots + u_n x_n' = 0$$

of P^n by homogeneous coordinates $u_0, \ldots, u_n$, written as a row vector $u = (u_0, \ldots, u_n)$. A cotangent vector at x is determined by the equation of a hyperplane u incident with x, $ux = 0$; if x is replaced by λx, u must be replaced by $u\lambda^{-1}$. Thus B may be described by (x, u), $ux = 0$, with $(\lambda x, u\lambda^{-1})$ equivalent to (x, u). M is consequently described by incident points and hyperplanes (x, u), $ux = 0$; now $(\lambda x, u\mu^{-1})$ is equivalent to (x, u). The Pfaffian form

$$\omega = udx = u_0 dx_0 + \ldots + u_n dx_n$$

is well-defined on B and gives the contact structure on the space M of co-directions, that is, on the space of hypersurface elements, of P^n [6, §63, p. 242].

A projectivity of P^n, a transformation in $PSL(n+1; \mathbf{C})$, which will be represented by $x \to gx$ with g in $SL(n+1; \mathbf{C})$, transforms hyperplanes by $u \to ug^{-1}$ and cotangent vectors and co-directions by $(x, u) \to (gx, ug^{-1})$. Since

$$g^*\omega = g^*(udx) = ug^{-1} d(gx) = ug^{-1} g dx = udx = \omega,$$

projectivities are contact transformations of M. In addition, since $ux = 0$, $udx = -{}^t x {}^t du$ and, hence, classical projective duality $(x, u) \to ({}^t u, -{}^t x)$ preserves ω and is a contact transformation [6, §62].

2.6. Let M be a complex contact manifold which is *algebraic*; M is a smooth algebraic variety and the contact structure is given by a bundle $B \to M$ and Pfaffian form ω on B which are algebraic.

Assume further that M is *connected, compact,* and *homogeneous* under a *linear algebraic group* G of contact automorphisms. Since M is connected, we may assume G is *connected.* We may also assume that G acts *effectively* on M: only the identity element of G acts as the identity transformation on M.

Now G is *semi-simple* [1, §4]. For the radical of G, a normal solvable subgroup, has a fixed point in compact M [3, (10.4)] and since, G acts effectively on M, this radical is trivial. Thus M is exhibited as G/P, with G connected and semi-simple, and P the isotropy subgroup of a point x_0 in M. Since G/P is compact, P is a *parabolic* subgroup of G [3, (11.2); 4, §68 ff.; and 8, §2]. P is its own normalizer in G, so contains the center of G; since G acts effectively on M, this center is trivial. G is *centerless.*

Since G is a linear algebraic group, we will throughout view the elements of it and its Lie algebra $\mathfrak{g}$ as matrices. Thus: For g in G and X in $\mathfrak{g}$, $Ad(g)\,X = gXg^{-1}$, a product of matrices. Left-invariant Pfaffian forms on G are given by $\omega_0\,(g^{-1}dg)$, where ω_0 is a linear function on $\mathfrak{g}$, and dg is the matrix of differentials of the entries of g. The action of $Ad\,(g)$ on left-invariant Pfaffian forms on G, i.e., $Ad\,(g)^*$, is then $({}^tAd(g)\omega_0)\,(X) = \omega_0\,(gXg^{-1})$.

2.7. From 2.3, G acts on the principal $\mathbf{C}^*$-bundle B over $M = G/P$. Let b_0 in B lie over the point x_0 in M fixed by P. If g is in P, then gb_0 lies over x_0, so $gb_0 = R_a b_0 = b_0 a$ for a unique $a = \chi\,(g)$ in $\mathbf{C}^*$. $\chi\colon P \to \mathbf{C}^*$ is a homomorphism. χ is either surjective or trivial, and in the former case G is transitive on B since it is then transitive on M and on the fibers of B over M. In fact, χ *is* surjective [2, §2]; the key lemma of Boothby's argument [2, p. 277] may be replaced by: The centralizer in $\mathfrak{g}$ of a nonzero element of $\mathfrak{g}$ is never a parabolic subalgebra. Thus B is exhibited as G/P_1 with P_1, the kernel of χ, a subgroup of P. The bundle $B \to M$ is $G/P_1 \to G/P$ with fiber $P/P_1 \simeq \mathbf{C}^*$.

By means of the map $G \to G/P_1$, pull the Pfaffian form ω on $B = G/P$, which defines the contact structure, up to a left-invariant form $\omega_0\,(g^{-1}dg)$ on G. This form is *$Ad\,(P_1)$-invariant*:

$$\omega_0\,(gXg^{-1}) = \omega_0\,(X)\,, \quad g \text{ in } P_1\,, \quad X \text{ in } \mathfrak{g}\,.$$

Let $\mathfrak{p}$ and $\mathfrak{p}_1$ denote the Lie algebras of P and P_1, respectively. Conditions (a, b, c) of 2.3 become:

(a) $(d\omega_0)^n \neq 0$;

(b) $\omega_0(X) = 0$, X in $\mathfrak{p}$; and

(c) $\omega_0(g^{-1}Xg) = \chi(g)\,\omega_0(X)$, g in P, X in $\mathfrak{g}$;

where $d\omega_0(X, Y) = -\frac{1}{2}\,\omega_0([X, Y])$ [1, (5.1), (5.2), (5.3)].

Since $\mathfrak{g}$ is semi-simple, its Killing form is non-degenerate and we may write

$$\omega_0(X) = \langle W, X \rangle, \ X \text{ in } \mathfrak{g},$$

for a unique W in $\mathfrak{g}$. Conditions (a, b, c) now become:

(a′) the centralizer of W in $\mathfrak{g}$ is $\mathfrak{p}_1$;

(b′) $\langle W, X \rangle = 0$, X in $\mathfrak{p}$; and

(c′) $[X, W] = \chi'(X)\,W$, X in $\mathfrak{p}$;

where χ' is the derivative of χ at the identity of P [1, (5.6)].

As a consequence of (c′), $\rho = \chi'$ restricted to a Cartan subalgebra contained in $\mathfrak{p}$ is a root of $\mathfrak{g}$; $E_\rho = W$ may be taken as the corresponding root vector. When the roots of $\mathfrak{g}$ are ordered, ρ is a positive root and $\rho + \alpha$, α a positive root, is not a root [1, (6.2)]. Hence ρ is the *maximal* root for this ordering and G is *simple* [1, (6.3) and 4, (25.6)].

2.8 Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, and choose a system of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Designate a subset of the simple roots as free and call the remaining simple roots non-free. An arbitrary root is called *free* if it contains a free simple root as a summand, and *non-free* if all its summands are non-free simple roots. A free root is necessarily positive. Besides free and non-free roots there are only the negatives of free roots [4, (69.23)]. If $\mathfrak{g}_\alpha$ denotes the root space for the root α, then

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \text{ non-free}} \mathfrak{g}_\alpha + \sum_{\alpha \text{ free}} \mathfrak{g}_\alpha$$

is a parabolic subalgebra of $\mathfrak{g}$, that is, it corresponds to a parabolic subgroup of any connected complex Lie group G having $\mathfrak{g}$ as its Lie algebra. Now, any parabolic subalgebra of $\mathfrak{g}$ is $Ad(G)$-conjugate to a parabolic subalgebra given by the above construction. Thus, once $\mathfrak{h}$ and the system of simple roots are fixed, the subsets of the simple roots classify parabolic subalgebras up to conjugacy [8, §2].

2.9 Continuing 2.7: We may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{p}$, choose a system of simple roots, and find the subset of free simple roots so that $\mathfrak{p}$ is given by the construction of 2.8 [8, §2]. The free and non-free roots are completely determined by the maximal root ρ, in that

$$\langle H_\rho, H_\alpha \rangle > 0 \text{ for } \alpha \text{ free,}$$
$$\langle H_\rho, H_\alpha \rangle = 0 \text{ for } \alpha \text{ non-free,}$$

where H_α in $\mathfrak{h}$ is defined by $\langle H_\alpha, H \rangle = \alpha(H)$, H in $\mathfrak{h}$ [1, (6.5)]. Consequently, we may describe $\mathfrak{p}$, $\mathfrak{p}_1$, and ω_0 for an algebraic homogeneous contact manifold in terms of the maximal root ρ by

(i) $\mathfrak{p} = \mathfrak{h} + \sum_{\langle H_\rho, H_\alpha \rangle \geqq 0} \mathfrak{g}_\alpha$,

(ii) $\mathfrak{p}_1$ = elements X of $\mathfrak{p}$ orthogonal to H_ρ, $\langle H_\rho, X \rangle = 0$, and

(iii) $\omega_0(X) = \langle E_\rho, X \rangle$, X in $\mathfrak{g}$,

[7, p. 1035]. Since G is connected and centerless with Lie algebra $\mathfrak{g}$, the groups G, P, P_1 and the form ω are completely determined.

2.10. Conversely, begin with a simple complex Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}$ and a system of simple roots. Using the maximal root ρ, define $\mathfrak{p}$, $\mathfrak{p}_1$, and ω_0 as in (i, ii, iii) of 2.9. Take for G the adjoint group of $\mathfrak{g}$, which is connected, centerless, and simple, and for P and P_1 the subgroups corresponding to $\mathfrak{p}$ and $\mathfrak{p}_1$. Then, $\omega_0(g^{-1}\,dg)$ is a left-invariant, $Ad(P_1)$-invariant Pfaffian form on G, and defines a form ω on G/P_1. The map $X \to \langle H_\rho, X \rangle$, X in $\mathfrak{p}$, gives rise to a homomorphism $\chi\colon P \to \mathbf{C}^*$. The form ω on the principal $\mathbf{C}^*$-bundle G/P_1 over G/P satisfies (a, b, c) of 2.3 and hence defines a contact structure making G/P a compact homogeneous algebraic contact manifold [1, Th. C and 7, p. 1035].

In this manner, Boothby established that there is exactly one compact homogeneous algebraic contact manifold, up to isomorphism, for each type A_n, ..., G_2 of simple complex Lie algebra [1, (7.1)]. For these manifolds, the group G is the identity component of the group of all contact automorphisms [7, (2.5)]. Boothby's classification [1, 2] was obtained with the assumptions that the complex contact manifold was compact, simply connected, and homogeneous under a complex Lie group of contact transformations, and used H. C. Wang's theory of compact homogeneous complex manifolds rather than parabolic subgroups. We may conclude that these contact manifolds are algebraic.

2.11. Let G be a semi-simple complex Lie group and G_0 a real form of G: G_0 is the set of elements of G fixed under a complex conjugation $g \to \bar{g}$. We use a bar to denote the conjugate of an object, and the terms *real* and *stable* refer to the conjugation.

Let P be a parabolic subgroup of G, and $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{p}$ the Lie algebras of G, G_0, P. $\mathfrak{g}_0 \cap \mathfrak{p}$ contains a stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\bar{\mathfrak{h}} = \mathfrak{h}$ [8, (2.6)]. If α is a root of $\mathfrak{g}$, so is $\bar{\alpha}$; α is real if $\bar{\alpha} = \alpha$. Now, choose a system of simple roots and find the subset of free simple roots so that $\mathfrak{p}$ is given by the construction of 2.8. Set $P_0 = G_0 \cap P$; G_0/P_0 is a subset of G/P. Wolf has shown that the following are *equivalent:*

(i) real dimension of G_0/P_0 = complex dimension of G/P,

(ii) G_0/P_0 is closed in G/P,

(iii) the set of free roots is stable,

(iv) $\mathfrak{p}$ or P is stable, and

(v) the algebraic manifold G/P is defined over $\mathbf{R}$ and G_0/P_0 is its set of real points,

[8, (3.6)]. When these conditions hold, G_0/P_0 is the unique closed orbit of G_0 on G/P [8, (3.4)]. We call G_0/P_0 a *real form* of G/P.

Let M be a compact, algebraic homogeneous contact manifold. Assume that M and its contact structure are *defined over* $\mathbf{R}$, that is, the principal $\mathbf{C}^*$-bundle $B \to M$ and the Pfaffian form ω on B are defined over $\mathbf{R}$. Let P be the isotropy subgroup of a real point x_0 of M in the group G of contact automorphisms. Then the complex conjugation on M defines one on G, $\bar{g}x = \overline{g\bar{x}}$, under which P is stable. Hence, we obtain a real form G_0 of G so that the real points of M are G_0/P_0, $P_0 = G_0 \cap P$. That ω is defined over $\mathbf{R}$ means $W = E_\rho$ lines in $\mathfrak{g}_0$, and the maximal root ρ is *real.* This is consistent with the stability of the set of free roots, as $\langle H_\rho, H_{\bar{\alpha}} \rangle > 0$ when $\langle H_\rho, H_\alpha \rangle > 0$. Consequently, the real forms of M correspond to the conjugations of $\mathfrak{g}$ for which ρ is real.

2.12. The method by which the contact structure on G/P will be exhibited, in the next sections, in classical form 2.2 is the following.

Let

$$\mathfrak{m} = \sum_{\langle H_\rho, H_\alpha \rangle < 0} \mathfrak{g}_\alpha ;$$

$\mathfrak{m}$ is supplementary to $\mathfrak{p}$ in $\mathfrak{g}$ and of dimension $2n-1$. We will determine X near 0 in $\mathfrak{m}$ as a function of $x_1, \ldots, x_n, p_1, \ldots, p_{n-1}$ so that $X \to (\exp X) \cdot x_0$

is one-to-one on an open neighborhood U of x_0 in G/P and $(\exp X) \cdot x_0$ is identifiable as the point $(x_1, \ldots, x_n)$ and the incident hyperplane

$$x_n' - x_n = p_1 (x_1' - x_1) + \ldots + p_{n-1} (x_{n-1}' - x_{n-1}) .$$

Now, $(\exp X) \cdot x_0 \to (\exp X) \cdot b_0$ is a section of the bundle G/P_1 over U and, via this section, the form ω on G/P_1 pulls down to

$$\omega_0 ((\exp X)^{-1} d (\exp X))$$

which, when expressed in terms of $x_1, \ldots, x_n, p_1, \ldots, p_{n-1}$, will be identified with

$$dx_n - p_1 dx_1 - \ldots - p_{n-1} dx_{n-1}$$

up to a constant multiple $a \neq 0$. For this latter calculation we will use

$$\begin{aligned} (\exp X)^{-1} d (\exp X) &= \frac{1 - e^{-ad\,X}}{ad\,X} (dX) \\ &= dX - \frac{1}{2} [X, dX] + \frac{1}{6} [X, [X, dX]] - \ldots \end{aligned}$$

[4, (10.2)], a series which is finite since $\mathfrak{m}$ is nilpotent. In fact, our choice of X will make the series for $\exp X$ themselves finite. The constant $a \neq 0$ could be made unity by using instead the section $(\exp X) \cdot x_0 \to (\exp X) g^{-1} \cdot b_0$, where g in P is chosen so that $\chi\,(g) = a$. This amounts to following the original section by R_a^{-1} in the bundle.

3. Co-directions in projective space

The contact structure on the $(2n-1)$-dimensional space of co-directions in complex projective space P^n, described in 2.5, is obtained when the construction of 2.10 is carried out for the simple complex Lie algebra of type A_n, $n \geqslant 1$.

3.1 Let $\mathfrak{g} = \mathfrak{sl}\,(n + 1;\ \mathbf{C})$, complex $(n+1)$ by $(n+1)$ matrices of trace zero. For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take the diagonal matrices of $\mathfrak{g}$. Let δ_i, $i = 0,1, \ldots, n$ be the linear function on $\mathfrak{h}$ which assigns to $H = \operatorname{diag}(h_1, \ldots, h_n)$ in $\mathfrak{h}$ the i^{th} diagonal element: $\delta_i (H) = h_i$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are

$$\delta_i - \delta_j \quad i,j = 0, 1, \ldots, n \text{ and } i \neq j$$

and the root vector E_α corresponding to the root α is

$$E_{\delta_i-\delta_j} = E_{ij}\,,$$

the matrix with 1 in the i^{th} row and j^{th} column and 0*s* elsewhere [4, (16.2)]. A system of simple roots is

$$\delta_0 - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\,,$$

for which the maximal root is

$$\rho = (\delta_0 - \delta_1) + (\delta_1 - \delta_2) + \ldots + (\delta_{n-1} - \delta_n) = \delta_0 - \delta_n$$

[4, App., Table *E*]. The Killing form of $\mathfrak{g}$ is $\langle X, Y \rangle = 2(n+1)\,\mathrm{tr}\,(XY)$, but we replace this with $\langle X, Y \rangle = \mathrm{tr}\,(XY)$ for convenience. Then the H_α in $\mathfrak{h}$ are given by

$$H_{\delta_i-\delta_j} = \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$$

with 1 and -1 in the i^{th} and j^{th} entry, respectively. Especially,

$$H_\rho = \mathrm{diag}\,(1, 0, \ldots, 0, -1)\,.$$

We have

$$\langle H_\rho, H_{\delta_i-\delta_j} \rangle \begin{cases} < 0 & j = 0 \text{ or } i = n \\ \geqslant 0 & \text{otherwise} \end{cases},$$

so that $\mathfrak{p}$ in (i) of 2.9 consists of matrices of the form

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & * & \cdots & * \\ 0 & \cdots & 0 & * \end{bmatrix}$$

of trace zero, where the starred entries are arbitrary.

3.2 The connected centerless simple group $G = PSL\,(n+1; \mathbf{C}) = SL\,(n+1; \mathbf{C})/\{\text{center}\}$ is transitive on the space consisting of points x and incident hyperplanes u, $ux = 0$, in P^n, as in 2.5. The isotropy subgroup P of the incident point and hyperplane

$$x_0 = {}^t(1, 0, \ldots, 0)\,, \quad u_0 = (0, \ldots, 0, 1)$$

has exactly $\mathfrak{p}$ for its Lie algebra. Hence, the homogeneous contact manifold which the construction of 2.10 gives is

$$\begin{aligned} G/P &= \text{space of incident points and hyperplanes in } P^n \\ &= \text{space of co-directions in complex } P^n. \end{aligned}$$

3.3 Let $\mathfrak{m}$ be the $(2n-1)$-dimensional supplement to $\mathfrak{p}$ in $\mathfrak{g}$ consisting of matrices of the form

$$\begin{bmatrix} 0 & & & \\ * & & 0 & \\ \vdots & & & \\ * & \cdots & * & 0 \end{bmatrix};$$

cf. 2.12. The product of any two matrices of $\mathfrak{m}$ has a nonzero entry only in the n^{th} row and 0^{th} column; the product of any three is zero. Set

$$X = \begin{bmatrix} 0 & & & \\ x_1 & & 0 & \\ \vdots & & & \\ x_{n-1} & & & \\ x_n - \frac{1}{2}\sum p_i x_i & p_1 \cdots p_{n-1} & & 0 \end{bmatrix},$$

where the summation is over $i = 1, 2, \ldots, n-1$. X is in $\mathfrak{m}$ and

$$\exp X = 1_{n+1} + X + \frac{1}{2}X^2 = \begin{bmatrix} 1 & & & \\ x_1 & \ddots & 0 & \\ \vdots & & \ddots & \\ x_{n-1} & 0 & & \\ x_n & p_1 \cdots p_{n-1} & & 1 \end{bmatrix},$$

$$(\exp X)^{-1} = 1_{n+1} - X + \frac{1}{2}X^2 = \begin{bmatrix} 1 & & & & \\ -x_1 & \ddots & & 0 & \\ \vdots & & \ddots & & \\ -x_{n-1} & & 0 & \ddots & \\ -x_n + \sum p_i x_i & -p_1 & \dots & -p_{n-1} & 1 \end{bmatrix}.$$

The point

$$x = (\exp X)\cdot x_0 = {}^t(1, x_1, \dots, x_n)$$

is incident with the hyperplane

$$u = u_0 \cdot (\exp X)^{-1} = (-x_n + \sum p_i x_i, \; -p_1, \dots, -p_{n-1}, 1),$$

and the hyperplane $ux' = 0$, $x' = {}^t(1, x_1', \dots, x_n')$, is

$$x_n' - x_n = p_1(x_1' - x_1) + \dots + p_{n-1}(x_{n-1}' - x_{n-1}).$$

Thus, this choice of X establishes the classically identifiable coordinates $x_1, \dots, x_n, p_1, \dots, p_{n-1}$ on G/P.

3.8 From $\rho = \delta_0 - \delta_n$, we have $W = E_\varrho = E_{0n}$ in (iii) of 2.9 and $\omega_0(X) = \langle W, X \rangle$ is the $n0$-entry of X. The form ω on G/P is obtained as $\omega = \omega_0((\exp X)^{-1} d(\exp X))$ with

$$(\exp X)^{-1} d(\exp X) = dX - \frac{1}{2}[X, dX]$$

as in 2.12. For X as in 3.3, the only nonzero entry in $[X, dX]$ is the $n0^{th}$ and it is $\sum p_i\, dx_i = -\sum x_i\, dp_i$. Hence

$$(\exp X)^{-1} d(\exp X) = \begin{bmatrix} 0 & & & & \\ dx_1 & & & 0 & \\ \vdots & & & & \\ dx_{n-1} & & & & \\ dx_n - \sum p_i dx_i & dp_1 & \dots & dp_{n-1} & 0 \end{bmatrix},$$

and the $n0$-entry is

$$\omega = dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1} \,.$$

This identifies the contact structure with the classical one as in 2.12.

3.5 The real contact structure on the $(2n-1)$-dimensional space of co-directions in real projective space P^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the connected centerless group $PSL(n+1; \mathbf{R})$ consisting of real contact automorphisms.

4. Higher sphere geometry

4.1 In complex Euclidean space E^n, the equation

$$x_1'^2 + \dots + x_n'^2 - 2a_1 x_1' - \dots - 2a_n x_n' + C = 0$$

describes a sphere with center $(a_1, \dots, a_n)$ and complex radius r given by

$$r^2 = a_1^2 + \dots + a_n^2 - C \,.$$

When $r \neq 0$, the two choices of sign for r is said to give two "orientations" to the sphere. Thus, the $n+2$ coordinates $a_1, \dots, a_n, r, C$, which are related by

$$a_1^2 + \dots + a_n^2 - r^2 - C = 0 \,,$$

describe the space of oriented spheres in E^n [6, §25].

Introduce homogeneous coordinates by

$$a_i = \frac{\alpha_i}{\nu} \,, \quad r = \frac{\lambda}{\nu} \,, \quad C = \frac{\mu}{\nu} \,,$$

$i = 1, 2, \dots, n$. Then the oriented spheres of E^n correspond to certain points of the quadric Ψ^{n+1} in P^{n+2} described by

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu\nu = 0 \,.$$

The sphere corresponding to the point $(\alpha_1, \dots, \alpha_n, \lambda, \mu, \nu)$ of Ψ^{n+1} is

$$\nu(x_1'^2 + \dots + x_n'^2) - 2\alpha_1 x_1' - \dots - 2\alpha_n x_n' + \mu = 0 \,.$$

Ordinary spheres have finite nonzero radius r, so $\nu \neq 0$. For $\nu = 0$, we obtain oriented hyperplanes. For $\lambda = 0$, we obtain point spheres or hyperplanes with isotropic hyperplane coordinate vector; these carry no orientation. If we include these special cases as spheres of E^n, then Ψ^{n+1} is the space of all oriented spheres in E^n.

Two spheres in E^n with centers $(a_1, ..., a_n)$, $(a_1', ..., a_n')$ and radii r, r' respectively are tangent, orientations taken into account, if

$$(a_1 - a_1')^2 + ... + (a_n - a_n')^2 = (r - r')^2 .$$

Use $a_1^2 + ... + a_n^2 = r^2 + C$ for both spheres to obtain the condition for tangency as

$$2a_1 a_1' + ... + 2a_n a_n' - 2rr' - C - C' = 0$$

or, in homogeneous coördinates,

$$2\alpha_1 \alpha_1' + ... + 2\alpha_n \alpha_n' - 2\lambda\lambda' - \mu\nu' - \nu\mu' = 0 .$$

Hence, two spheres of E^n are tangent when their corresponding points in Ψ^{n+1} are conjugate, that is, the line joining these points lies entirely in Ψ^{n+1} [6, §25].

4.2. A pencil of mutually tangent spheres in E^n corresponds to a line in Ψ^{n+1}. This pencil of spheres determines an "oriented complex co-direction" in E^n since it contains a point sphere and an incident oriented hyperplane. Corresponding to the hyperplane

$$x_n' - x_n = p_1 (x_1' - x_1) + ... + p_{n-1} (x_{n-1}' - x_{n-1})$$

at the point $(x_1, ..., x_n)$ is the line

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hline 0 \\ xx \\ 1 \end{bmatrix} + t \begin{bmatrix} -p_1 \\ \vdots \\ -p_{n-1} \\ 1 \\ \hline -\sqrt{pp+1} \\ 2(x_n - px) \\ 0 \end{bmatrix}$$

of Ψ^{n+1}, where

$$xx = \sum_{i=1}^{n} x_i^2, \quad px = \sum_{i=1}^{n-1} p_i x_i, \quad pp = \sum_{i=1}^{n-1} p_i^2;$$

this is the pencil of spheres

$$\sum_{i=1}^{n-1} (x_i' - x_i + tp_i)^2 + (x_n' - x_n - t)^2 = t^2 \left(\sum_{i=1}^{n-1} p_i^2 + 1 \right)$$

passing through $(x_1, ..., x_n)$ and having their centers on the line normal to the hyperplane at this point.

For later calculations it will be convenient to replace $-p, ..., -p_{n-1}, 1$ by homogeneous $u_1, ..., u_{n-1}, u_n$. The line in Ψ^{n+1} corresponding to the hyperplane

$$u_1 (x_1' - x_1) + \ldots + u_n (x_n' - x_n) = 0$$

at the point $(x_1, ..., x_n)$ is then

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hdashline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hdashline 0 \\ xx \\ 1 \end{bmatrix} + t \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ u_n \\ \hdashline -\sqrt{uu} \\ 2ux \\ 0 \end{bmatrix},$$

where

$$xx = \sum_{i=1}^{n} x_i^2, \quad ux = \sum_{i=1}^{n} u_i x_i, \quad uu = \sum_{i=1}^{n} u_i^2.$$

Any convenient condition may be imposed on uu.

4.3. The contact structure on the $(2n-1)$-dimensional space of lines in Ψ^{n+1}, that is, the space of oriented co-directions in complex Euclidean space E^n, is obtained when the construction of 2.10. is carried out for the simple complex Lie algebra of type B_l or D_l, $l \geqslant 2$ and $l \geqslant 3$ respectively. However, it will be simpler to identify quantities geometrically if we proceed by using the description of 2.7, since now the groups are determined first.

Let

$$A = \left[\begin{array}{c|ccc} 2\cdot 1_n & & 0 & \\ \hline & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & 0 & -1 & 0 \end{array}\right]$$

be the matrix of the quadratic form defining Ψ^{n+1} in P^{n+1}. $SO(A; \mathbf{C})$, the special orthogonal group of this form, consists of matrices g in $SL(n+3; \mathbf{C})$ for which ${}^t gAg = A$. The connected centerless simple group $G = PSO(A; \mathbf{C}) = SO(A; \mathbf{C})/\{\text{center}\}$ is transitive on the lines of Ψ^{n+1} by Witt's theorem. Let l_0 be the line

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hline -1 \\ 0 \\ 0 \end{bmatrix}$$

of Ψ^{n+1}, joining

$${}^t(0, \ldots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad {}^t(0, \ldots, 0, 1 \mid -1, 0, 0)\,;$$

this corresponds to the pencil of spheres

$$\sum_{i=1}^{n-1} x_i'^2 + (x_n' - t)^2 = t^2$$

tangent to the hyperplane $x_n = 0$ at the origin of E^n, suitably oriented, as in 4.2. Let P denote the isotropy subgroup of l_0. Then

$$\begin{aligned} G/P &= \text{space of lines in } \Psi^{n+1} \\ &= \text{space of pencils of mutually tangent oriented spheres in } E^n \\ &= \text{space of oriented co-directions in complex } E^n. \end{aligned}$$

The Lie algebra $\mathfrak{g}$ of G consists of $(n+3)$ by $(n+3)$ matrices X for which ${}^tXA + AX = 0$. The matrices of $\mathfrak{g}$ are of the form

$$\left[\begin{array}{ccc|ccc}
\multicolumn{3}{c|}{\text{n by n skew-symmetric}} & b_1 & c_1 & d_1 \\
 & & & \vdots & \vdots & \vdots \\
 & & & b_{n-1} & c_{n-1} & d_{n-1} \\
 & & & b_n & c_n & d_n \\ \hline
b_1 \ldots & b_{n-1} & b_n & 0 & c & d \\
2d_1 \ldots & 2d_{n-1} & 2d_n & -2d & e & 0 \\
2c_1 \ldots & 2c_{n-1} & 2c_n & -2c & 0 & -e
\end{array}\right].$$

P consists of those elements of G which send the subspace of $\mathbf{C}^{n+3}$ spanned by

$${}^t(0, \ldots, 0, 0 \,|\, 0, 0, 1) \quad \text{and} \quad {}^t(0, \ldots, 0, 1 \,|\, -1, 0, 0)$$

into itself; the Lie algebra $\mathfrak{p}$ of P consists of those elements of $\mathfrak{g}$ which do the same. Hence, the matrices of $\mathfrak{p}$ are of the form

$$\left[\begin{array}{ccc|ccc}
\multicolumn{2}{c}{(n-1)\text{ by }(n-1)} & b_1 & b_1 & c_1 & 0 \\
\multicolumn{2}{c}{\text{skew-symmetric}} & \vdots & \vdots & \vdots & \vdots \\
 & & b_{n-1} & b_{n-1} & c_{n-1} & 0 \\
-b_1 \ldots & -b_{n-1} & 0 & b_n & c_n & -d \\ \hline
b_1 \ldots & b_{n-1} & b_n & 0 & c & d \\
0 \ldots & 0 & -2d & -2d & e & 0 \\
2c_1 \ldots & 2c_{n-1} & 2c_n & -2c & 0 & -e
\end{array}\right].$$

Note that $\mathfrak{g}$ and $\mathfrak{p}$ have dimensions $\frac{1}{2}(n+3)(n+2)$ and $\frac{1}{2}(n-1)(n-2) + 2n + 3 = \frac{1}{2}(n+3)(n+2) - 2n + 1$, respectively, in agreement with G/P having dimension $2n-1$.

4.4. For $n \geqslant 2$, set $n+3 = 2l+1$ or $2l$ according as n is even or odd. $\mathfrak{g}$ is of type B_l or D_l, $l \geqslant 2$ and $l \geqslant 3$ respectively.

For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take matrices of the form

$$H = \operatorname{diag}\left[\begin{array}{c|c} 0 & \begin{bmatrix} 0 & h_1 \\ -h_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & h_{l-2} \\ -h_{l-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{bmatrix} \end{array}\right];$$

the first row and column occur only in case B_l, it is suppressed for case D_l. The Killing form of $\mathfrak{g}$ is $\langle X, Y \rangle = (n+1) \operatorname{tr}(XY)$, but we replace this with $\langle X, Y \rangle = \frac{1}{2} \operatorname{tr}(XY)$ for convenience.

Let W in $\mathfrak{p}$ be

$$W = \left[\begin{array}{c|cc|ccc} 0 & 0 & & & 0 & \\ \hline 0 & 0 & & & 0 & \\ & & 0 & 0 & -\frac{1}{2} & 0 \\ \hline & & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & & & & \\ & & 0 & 0 & 0 & 0 \\ & & -1 & -1 & 0 & 0 \end{array}\right]$$

For H in $\mathfrak{h}$ we have $[H, W] = -(h_{l-1}+h_l)\,W$, so $\rho(H) = -(h_{l-1}+h_l)$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $W = E_\rho$ is the corresponding root vector.

For X in $\mathfrak{g}$ as described in 4.3, direct calculation shows $[X, W] = 0$ implies X is in $\mathfrak{p}$ and $b_n + e = 0$; thus the centralizer of W in $\mathfrak{g}$ consists of those elements of $\mathfrak{p}$ with $b_n + e = 0$. For X in $\mathfrak{p}$ now, the same calculation gives $[X, W] = -(b_n+e)\,W$, so $[X, W] = \rho(X)\,W$ with ρ extended to $\mathfrak{p}$ by $\rho(X) = -(b_n+e)$. Finally, W is orthogonal to $\mathfrak{p}$ with respect to the Killing form. Hence, (a', c', b') of 2.7 are satisfied, and W is the element of $\mathfrak{g}$ giving the contact structure on G/P.

The origin of the element W is not immediately evident. It was obtained by determining the maximal root and corresponding root vector for Lie algebras of type B_l and D_l when the quadratic form is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \ldots + 2\xi_l\xi_{2l}$$

and then passing to the form

$$\alpha_1^2 + \ldots + \alpha_n^2 - \lambda^2 - \mu\nu$$

by conjugating by the element of $PSL(n+3; \mathbf{C})$ which corresponds to the "line-sphere transformation". This will be described further in the next section.

4.5. Let $\mathfrak{m}$ be the $(2n-1)$-dimensional supplement to $\mathfrak{p}$ in $\mathfrak{g}$ consisting of matrices of the form

$$\left[\begin{array}{ccc|ccc}
 & & -b_1 & b_1 & 0 & d_1 \\
 & 0 & \vdots & \vdots & \vdots & \vdots \\
 & & -b_{n-1} & b_{n-1} & 0 & d_{n-1} \\
b_1 \ldots & b_{n-1} & 0 & 0 & 0 & d_n \\
\hline
b_1 \ldots & b_{n-1} & 0 & 0 & 0 & d_n \\
2d_1 \ldots & 2d_{n-1} & 2d_n & -2d_n & 0 & 0 \\
0 \ldots & 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

cf. 2.12. For X in $\mathfrak{m}$ we have

$$X^2 = \left[\begin{array}{cc|ccc} 0 & & & 0 & \\ & -bb & bb & 0 & bd \\ \hline & -bb & bb & 0 & bd \\ 0 & -2bd & 2bd & 0 & 2dd \\ & 0 & 0 & 0 & 0 \end{array}\right].$$

where

$$bb = \sum_{i=1}^{n-1} b_i^2, \; bd = \sum_{i=1}^{n-1} b_i d_i, \; dd = \sum_{i=1}^{n-1} d_i^2 .$$

The product of any three matrices of $\mathfrak{m}$ is zero. Especially,

$$\exp X = 1_{n+3} + X + \frac{1}{2} X^2 .$$

In order to establish classically identifiable coordinates on G/P as in 2.12, we must determine X in $\mathfrak{m}$ so that $(\exp X) \cdot l_0$ is the line of Ψ^{n+1} described in 4.2. With X in $\mathfrak{m}$ as above, $(\exp X) \cdot l_0$ is the line joining the points

$$(\exp X) \cdot \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} d_1 \\ \vdots \\ d_{n-1} \\ d_n + \frac{1}{2} bd \\ \hline d_n + \frac{1}{2} bd \\ dd \\ 1 \end{array}\right]$$

and

$$(\exp X)\cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hdashline -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1-bb \\ \hdashline -1-bb \\ 4d_n - 2bd \\ 0 \end{bmatrix}.$$

On this line we can identify the point sphere when $\lambda = 0$, giving

$$\begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \\ d_n + \frac{1}{2}bd \\ \hdashline d_n + \frac{1}{2}bd \\ dd \\ 1 \end{bmatrix} + \frac{d_n + \frac{1}{2}bd}{1+bb} \begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1-bb \\ \hdashline -1-bb \\ 4d_n - 2bd \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hdashline 0 \\ xx \\ 1 \end{bmatrix},$$

and the incident oriented hyperplane when $\nu = 0$, giving

$$\begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1-bb \\ \hdashline -1-bb \\ 4d_n - 2bd \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ u_n \\ \hdashline -\sqrt{uu} \\ 2ux \\ 0 \end{bmatrix}.$$

These equations will be satisfied if be we impose the condition $\sqrt{uu} = 1 + bb$ on uu, or

$$u_i = -2b_i, \quad u_n = 1 - bb,$$

$i = 1, 2, \dots, n-1$, and then set

$$b_i = -\frac{1}{2} u_i,$$

$$d_i = x_i - \frac{1}{2} u_i x_n, \qquad i = 1, 2, \dots, n-1$$

$$d_n = \frac{1}{4} \sum_{i=1}^{n-1} u_i x_i + \frac{1}{2} x_n.$$

Thus, this choice of X establishes the classically identifiable coordinates $x_1, \dots, x_n, u_1, \dots, u_n$ on G/P as in 2.12 and 4.2.

4.6. From 2.12, the form ω on G/P is obtained as

$$\omega = \langle W, (\exp X)^{-1} d(\exp X) \rangle$$

with
$$(\exp X)^{-1} d(\exp X) = dX - \frac{1}{2}[X, dX].$$

Take X as in 4.5 and let the entries of dX be denoted as those of X with primes affixed. Then

$$(\exp X)^{-1}\, d\,(\exp X) \;=\; \left[\begin{array}{cc|ccc}
 & -b_1' & b_1' & 0 & d_1' \\
0 & \vdots & \vdots & \vdots & \vdots \\
 & -b'_{n-1} & b'_{n-1} & 0 & d_n{}'_{-1} \\
b_1' \dots b'_{n-1} & 0 & 0 & 0 & d_n' - \frac{1}{2}c \\
\hline
b_1' \;\dots\; b'_{n-1} & 0 & 0 & 0 & d_n' - \frac{1}{2}c \\
2d_1' \dots 2d'_{n-1} \;\; & 2d'_n - c & -2d_n' + c & 0 & 0 \\
0 \;\dots\; 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

where

$$c \;=\; \sum_{i=1}^{n-1} (b_i d_i' - d_i b_i'),$$

and consequently, from the definition of W in 4.4, $\omega \;=\; c - 2d_n'$. Using the expressions in 4.5 for $b_1, \dots, b_{n-1}, d_1, \dots, d_n$ in terms of $x_1, \dots, x_n, u_1, \dots, u_n$, we obtain

$$\omega \;=\; -\sum_{i=1}^{n-1} u_i dx_i \;-\; \left[1 - \frac{1}{4}\sum_{i=1}^{n-1} u_i^2\right] dx_n$$

or, since $1 - \dfrac{1}{4}\sum_{i=1}^{n-1} u_i^2 \;=\; u_n,$

$$\omega \;=\; -\,(u_1\, dx_1 + \dots + u_n\, dx_n)\,.$$

This identifies the contact structure with the classical one as in 2.12 and 4.2.

4.7. The real contact structure on the $(2n-1)$-dimensional space of oriented co-direction in real Euclidean space E^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the two-component centerless group $PSO\;(A;\, \mathbf{R})$ consisting of real contact automorphisms.

5. The line-sphere transformation

The homogeneous contact manifold of co-directions in complex projective space P^3, obtained from the simple complex Lie algebra of type A_3, must coinside with that of oriented co-directions in complex Euclidean space E^3, obtained from the algebra of type D_3, in view of the isomorphisms $A_3 \simeq D_3$. To exhibit this explicitly, we introduce a third homogeneous contact manifold in terms of which both of these can be conveniently described, namely, the space of lines in the quadric Ω^4 in P^5 of Section 1.

5.1. We carry out the construction of 2.10 for the simple complex Lie algebras of type B_l and D_l, making the restriction to type D_3 later.

Let $\mathfrak{g} = \mathfrak{o}(A; \mathbf{C})$, complex square matrices X for which ${}^tXA + AX = 0$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & 1_l & 0 \end{bmatrix} \quad \text{in case } B_l$$

or

$$A = \begin{bmatrix} 0 & 1_l \\ 1_l & 0 \end{bmatrix} \quad \text{in case } D_l \quad ,$$

that is, the quadratic form defining $\mathfrak{g}$ is

$$\xi_0{}^2 + 2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l}$$

or

$$2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l}$$

respectively [4, (16.3) and (16.4)].

We exhibit the details of the construction for the case of D_l. For B_l one need only carry along an additional initial row and column in the matrices, as well as the corresponding roots; the conclusions are the same.

Thus $\mathfrak{g}$ consists of $2l$ by $2l$ matrices of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{bmatrix},$$

where X_1 is l by l and arbitrary and X_2 and X_3 are l by l and skew-symmetric. For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take diagonal matrices H of the form

$$H = \operatorname{diag}(h_1, \ldots, h_l \mid -h_1, \ldots, -h_l).$$

Let δ_i, $i = 1, 2, \ldots, l$ be the linear function on $\mathfrak{h}$ which assigns h_i to H: $\delta_i(H) = h_i$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are

$$\pm\delta_i \pm \delta_j \qquad i, j = 1, 2, \ldots, l \quad \text{and} \quad i \neq j$$

and the root vector E_α corresponding to the root α is

$$E_{\delta_i-\delta_j} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix}, \quad i \neq j,$$

$$E_{\delta_i+\delta_j} = \begin{bmatrix} 0 & E_{ij}-E_{ji} \\ 0 & 0 \end{bmatrix}, \quad i < j,$$

$$E_{-\delta_i-\delta_j} = \begin{bmatrix} 0 & 0 \\ E_{ji}-E_{ij} & 0 \end{bmatrix}, \quad i < j,$$

where E_{ij} is the l by l matrix with 1 in the i^{th} row and j^{th} column and 0*s* elsewhere [4, (16.3)]. A system of simple roots is

$$\delta_1 - \delta_2,\ \delta_2 - \delta_3, \ldots, \delta_{l-1} - \delta_l, \quad \text{and} \quad -\delta_1 - \delta_2,$$

(this is not the same choice as in [4, (16.3)]), for which the maximal root is

$$\rho = -\delta_{l-1} - \delta_l,$$

[4, App., Table E]. The Killing form of $\mathfrak{g}$ is $\langle X, Y \rangle = (2l-2)\operatorname{tr}(XY)$, but we replace this with $\langle X, Y \rangle = \frac{1}{2}\operatorname{tr}(XY)$ for convenience. Then the H_α in $\mathfrak{h}$ are given by

$$H_{\pm\delta_i\pm\delta_j} = \operatorname{diag}(0, \ldots, 0, \pm 1, 0, \ldots, 0, \pm 1, 0, \ldots, 0 \mid \text{———}),$$

where the ± 1*s* occur in the i^{th} and j^{th} entries and the second l entries are the negatives of the first l entries. Especially,

$$H_\rho = \operatorname{diag}(0, \ldots, 0, -1, -1 \mid 0, \ldots, 0, 1, 1).$$

It is now straightforward to determine for which roots α we have $\langle H_\rho, H_\alpha \rangle \geqslant 0$ and find that $\mathfrak{p}$ in (i) of 2.9 consists of matrices of the form

$$\left[\begin{array}{c:cccccc}
 & \text{arbitrary} & & & & 0 & 0 \\
\text{negative} & (l-2) \text{ by } (l-2) & & & & \vdots & \vdots \\
\text{transpose} & \text{skew-} & & & & & \\
\text{of lower} & \text{symmetric} & & & & 0 & 0 \\
\text{right} & 0 & \cdots & & 0 & 0 & 0 \\
 & 0 & \cdots & & 0 & 0 & 0 \\
\hdashline
 & * & \cdots & & * & 0 & 0 \\
\text{arbitrary} & \vdots & & & \vdots & \vdots & \vdots \\
l \text{ by } l & & & & & & \\
\text{skew-} & & & & & 0 & 0 \\
\text{symmetric} & & & & & * & * \\
 & * & \cdots & & * & * & *
\end{array}\right],$$

where the starred entries are arbitrary.

5.2. The connected centerless simple group $G = PSO(A; \mathbf{C})$ is transitive on the lines of the quadric Ω^{2l-2}

$$\xi_1 \xi_{l+1} + \dots + \xi_l \xi_{2l} = 0$$

in P^{2l-1} by Witt's theorem. The Lie algebra of the isotropy subgroup of the line l_0 joining

$${}^t(0, \dots, 0, 1, 0) \quad \text{and} \quad {}^t(0, \dots, 0, 0, 1)$$

is $\mathfrak{p}$. Hence

$$G/P = \text{space of lines in } \Omega^{2l-2}.$$

The element $W = E_\rho$ of $\mathfrak{p}$ giving the contact structure on G/P, as in 2.7, is

$$W = \left[\begin{array}{c:c}
0 & 0 \\
\hdashline
\begin{array}{cc} & \\ & 0 \\ & \\ 0 & -1 \\ 1 & 0 \end{array} & 0
\end{array}\right]$$

In general, the construction of 2.10 gives the $(2n-1)$-dimensional homogeneous contact manifold of lines in the quadric Ω^{n+1} in P^{n+2}, where Ω^{n+1} is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \ldots + 2\xi_l\xi_{2l} = 0$$

in case B_l when n is even, $n+3 = 2l+1$, and Ω^{n+1} is Ω^{2l-2} above in case D_l when n is odd, $n+3 = 2l$; $n \geqslant 2$.

The real contact structure on the $(2n-1)$ dimensional space of lines of Ω^{n+1} in real projective space P^{n+2} is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the one- or two- component centerless group $PSO(A; \mathbf{R})$ consisting of real contact automorphisms.

5.3. The line joining $x = {}^t(x_0, x_1, x_2, x_3)$ and $y = {}^t(y_0, y_1, y_2, y_3)$ in complex projective space P^3 has Plücker coordinates $p_{ij} = x_iy_j - x_jy_i$. These coordinates are the coefficients of the bivector $x \wedge y$ with respect to the basis

$$e_1 \wedge e_2\,,\ e_3 \wedge e_1\,,\ e_2 \wedge e_3\,,\ e_0 \wedge e_3\,,\ e_0 \wedge e_2\,,\ e_0 \wedge e_1\,,$$

where $e_0 = {}^t(1, 0, 0, 0), \ldots, e_3 = {}^t(0, 0, 0, 1)$, and satisfy

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0\,,$$

[6, §69]. If we set

$$\xi_1 = p_{12}\,,\quad \xi_2 = p_{31}\,,\quad \xi_3 = p_{23}\,,$$
$$\xi_4 = p_{03}\,,\quad \xi_5 = p_{02}\,,\quad \xi_6 = p_{01}\,,$$

we have that the lines of P^3 correspond to the points of the quadric Ω^4

$$\xi_1\xi_4 + \xi_2\xi_5 + \xi_3\xi_6 = 0$$

in P^5. Two lines of P^3 intersect exactly when their corresponding points on Ω^4 are conjugate, that is, the line joining these points lies entirely in Ω^4.

To a point x in P^3 we associate all lines of P^3 incident with x and hence a plane lying in Ω^4. To a plane u in P^3 we associated all lines of P^3 lying in u and hence a plane lying in Ω^4. These two families of planes doubly rule Ω^4. To a surface element or co-direction in P^3, that is, a point x and incident plane u, is then associated all lines of P^3 lying in u and incident with x. In Ω^4 this corresponds to the intersection of the planes corresponding to x and u and is a line. Hence, the 5-dimensional spaces of co-directions in P^3 and lines in Ω^4 correspond naturally.

Note that the co-direction in P^3 consisting of the point $x_0 = {}^t(1, 0, 0, 0)$ and the incident plane $u_0 \colon x_3 = 0$ in 3.2 corresponds to the line l_0 of Ω^4 joining the points ${}^t(0, 0, 0, 0, 1, 0)$ and ${}^t(0, 0, 0, 0, 0, 1)$ in 5.2. For, to the co-direction (x_0, u_0) is associated all lines of P^3 joining x_0 and a point $y = {}^t(y_0, y_1, y_2, 0)$ of u_0; such a line has Plücker coordinates

$$\begin{aligned} &\xi_1 = 0\,, \quad \xi_2 = 0\,, \quad \xi_3 = 0\,, \\ &\xi_4 = 0\,, \quad \xi_5 = y_2\,, \quad \xi_6 = y_1\,, \end{aligned}$$

and corresponds to a point of Ω^4 lying on l_0.

The projectivity g in $PSL(4; \mathbf{C})$ permutes the lines of P^3 by $x \wedge y \to gx \wedge gy$, a projectivity of P^5 which preserves Ω^4. In this way one obtains the isomorphism $A_3 \simeq D_3$:

$$PSL(4; \mathbf{C}) \simeq PSO(A; \mathbf{C})\,, \quad A = \begin{bmatrix} 0 & 1_3 \\ 1_3 & 0 \end{bmatrix},$$

[4, (25.8.4′)]. The spaces of co-directions in P^3 and lines in Ω^4 are homogeneous under $PSL(4; \mathbf{C})$ and $PSO(\mathrm{A}; \mathbf{C})$ respectively; hence the correspondence between these spaces is as homogeneous spaces. In fact, since (x_0, u_0) and l_0 correspond, their isotropy subgroups, as described in 3.2 and 5.2, correspond under the isomorphism.

From the isomorphism of the groups, we obtain the isomorphism of the Lie algebras $\mathfrak{sl}(4; \mathbf{C}) \simeq \mathfrak{o}(A; \mathbf{C})$, where X in $\mathfrak{sl}(4; \mathbf{C})$ is sent to the linear transformation $x \wedge y \to (Xx) \wedge y + x \wedge (Xy)$ in $\mathfrak{o}(A; \mathbf{C})$. With $X = (a_{ij})$, $i, j = 0, 1, 2, 3$, the matrix of this transformation with respect to the basis $e_i \wedge e_j$ is

$$\left[\begin{array}{ccc|ccc} a_{11}+a_{22} & -a_{23} & -a_{13} & 0 & a_{10} & -a_{20} \\ -a_{32} & a_{11}+a_{33} & -a_{12} & -a_{10} & 0 & a_{30} \\ -a_{31} & -a_{21} & a_{22}+a_{33} & a_{20} & -a_{30} & 0 \\ \hline 0 & -a_{01} & a_{02} & a_{00}+a_{33} & a_{32} & a_{31} \\ a_{01} & 0 & -a_{03} & a_{23} & a_{00}+a_{22} & a_{21} \\ -a_{02} & a_{03} & 0 & a_{13} & a_{12} & a_{00}+a_{11} \end{array}\right];$$

this describes the isomorphism explicitly. Under this isomorphism, the Lie algebras of the isotropy subgroups of (x_0, u_0) and l_0, as in 3.1 and 5.1, correspond. Moreover, the element

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{of } \mathfrak{sl}(4; \mathbf{C})$$

is sent into the element

$$\left[\begin{array}{ccc|c} & 0 & & 0 \\ \hline 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \end{array}\right] \quad \text{of } \mathfrak{o}(A; \mathbf{C}).$$

Since these are the root vectors for the maximal roots which determine the contact structures, as in 3.4 and 5.2, we conclude:

The 5-dimensional manifolds of co-directions in P^3 and lines in Ω^4 are isomorphic as algebraic homogeneous contact manifolds.

This isomorphism holds for the real contact manifolds also; cf. 3.5 and 5.2. The real connected centerless groups $PSL(4; \mathbf{R})$ and $PSO(A; \mathbf{R})$ are isomorphic; each consists of the elements fixed under complex conjugation of matrix entries.

5.4. The algebraic homogeneous contact manifolds of lines in the quadrics Ψ^{n+1} and Ω^{n+1}, 4.3 and 5.2, are isomorphic since they are both obtained from the simple complex Lie algebra of type B_l or D_l by the construction of 2.10. This isomorphism can be exhibited explicitly by means of a contact transformation which reduces to the line-sphere transformation, as described in Section 1, when $n = 3$.

Throughout, unprimed quantities refer to Ω^{n+1} and primed quantities to Ψ^{n+1}. Set $n + 3 = 2l + 1$ or $2l$ according as n is even or odd; $n \geqslant 2$.

Thus,

$$G = PSO(A;\, \mathbf{C}), \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & 1_l & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1_l \\ 1_l & 0 \end{bmatrix}$$

and

$$G' = PSO(A';\, \mathbf{C}), \quad A' = \left[\begin{array}{c|ccc} 2\,.\,1_n & & 0 & \\ \hline & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & 0 & -1 & 0 \end{array}\right].$$

These are groups of projectivities preserving Ω^{n+1} and Ψ^{n+1}, respectively, in P^{n+2}.

In case n is odd, the transformation which we consider is

$$\begin{array}{rlrl} \xi_1 &= \alpha_1 + \sqrt{-1}\,\alpha_2 & \xi_{l+1} &= \alpha_1 - \sqrt{-1}\,\alpha_2 \\ \xi_2 &= \alpha_3 + \sqrt{-1}\,\alpha_4 & \xi_{l+2} &= \alpha_3 - \sqrt{-1}\,\alpha_4 \\ &\vdots & &\vdots \\ \xi_{l-2} &= \alpha_{n-2} + \sqrt{-1}\,\alpha_{n-1} & \xi_{2l-2} &= \alpha_{n-2} - \sqrt{-1}\,\alpha_{n-1} \\ \xi_{l-1} &= \alpha_n + \lambda & \xi_{2l-1} &= \alpha_n - \lambda \\ \xi_l &= \mu & \xi_{2l} &= -\nu\,. \end{array}$$

This is a projectivity of P^{n+2} which sends the quadric Ψ^{n+1}

$$\alpha_1^{\,2} + \dots + \alpha_n^{\,2} - \lambda^2 - \mu\nu = 0$$

into the quadric Ω^{n+1}

$$2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l} = 0\,.$$

In case n is even, the first equation of the transformation is $\xi_0 = \sqrt{2}\,\alpha_1$ and the remaining ones are like the above.

As before, we exhibit the details of the calculations for the case of n odd. For n even one need only carry along an additional initial row and column in the matrices; the conclusions are unchanged.

The matrix T of the transformation is

$$T = \begin{bmatrix} B & & & 0 & \\ & 1 & 1 & 0 & 0 \\ 0 & & & & \\ & 0 & 0 & 1 & 0 \\ \bar{B} & & & 0 & \\ & 1 & -1 & 0 & 0 \\ 0 & & & & \\ & 0 & 0 & 0 & -1 \end{bmatrix},$$

where B is the $(l-2)$ by $(2l-4)$ matrix

$$B = \begin{bmatrix} 1 \;\; \sqrt{-1} & & & \\ & 1 \;\; \sqrt{-1} & & 0 \\ & & \cdot & \\ 0 & & \cdot & \\ & & \cdot & \\ & & & 1 \;\; \sqrt{-1} \end{bmatrix}$$

and $\bar{B}$ is its complex conjugate; T has inverse

$$T^{-1} = \frac{1}{2} \begin{bmatrix} {}^t\bar{B} & 0 & & {}^tB & 0 & \\ & 1 & 0 & & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ & 0 & 2 & & 0 & 0 \\ & 0 & 0 & & 0 & -2 \end{bmatrix}.$$

By direct calculation we ascertain the following:

(1) $A' = {}^tTAT$ and hence $G' = T^{-1}GT$. G and G' are conjugate, but do not coincide, in $PSL(n+3; \mathbf{C})$. As a consequence, $\mathfrak{g}' = T^{-1}\mathfrak{g}T$.

(2) $l_0' = T^{-1}l_0$; the line l_0 in Ω^{n+1} joining

$${}^t(0, \ldots, 0, 1, 0) \quad \text{and} \quad {}^t(0, \ldots, 0, 0, 1)$$

is sent to the line l_0' of Ψ^{n+1} joining

$$^t(0, \ldots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad ^t(0, \ldots, 0, 1 \mid -1, 0, 0).$$

Hence their isotropy subgroups, as in 5.2 and 4.3 are conjugate: $P' = T^{-1} P T$. As a consequence, $\mathfrak{p}' = T^{-1} \mathfrak{p} T$.

(3) The Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$ in 5.1 and 4.4 are conjugate: $\mathfrak{h}' = T^{-1} \mathfrak{h} T$. In fact, for

$$H = \operatorname{diag}(h_1, \ldots, h_l \mid -h_1, \ldots, -h_l)$$

in $\mathfrak{h}$, we have

$$T^{-1}HT = \operatorname{diag}\left[\begin{bmatrix} 0 & \sqrt{-1}h_1 \\ -\sqrt{-1}h_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \sqrt{-1}h_{l-2} \\ -\sqrt{-1}h_{l-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{bmatrix}\right]$$

in $\mathfrak{h}'$.

(4) The elements W and W' of the Lie algebras which give the contact structures on G/P and G'/P', as in 5.2 and 4.4, are conjugate: $W' = T^{-1} W T$. We conclude:

The $(2n-1)$-dimensional manifolds of lines in Ω^{n+1} and lines in Ψ^{n+1} are isomorphic as algebraic homogeneous contact manifolds. The isomorphism is a consequence of the projectivity T carrying Ψ^{n+1} into Ω^{n+1}. T sends lines of Ψ^{n+1} into lines of Ω^{n+1} and is a contact transformation.

5.5. $G_0 = PSO(A; \mathbf{R})$ is a real form of G; it consists of the elements of G fixed under the conjugation $g \to \bar{g}$ of G, complex conjugation of the matrix entries of g. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, as in 5.1, is stable and the maximal root $\rho = -\delta_{l-1} - \delta_l$ is real. With $P_0 = G_0 \cap P$, we obtain from 2.11 the real contact manifold

$$G_0/P_0 = \text{space of lines in } \Omega^{n+1} \text{ in real } P^{n+2},$$

a real form of G/P; cf. 5.2. The same remarks apply to the real form $G_0' = PSO\,(A'; \mathbf{R})$ of G' for the conjugation $g' \to \bar{g}'$. With $P_0' = G_0' \cap P'$, we obtain the real contact manifold

$$
\begin{aligned}
G_0'/P_0' &= \text{space of lines in } \Psi^{n+1} \text{ in real } P^{n+2} \\
&= \text{space of pencils of mutually tangent oriented spheres in real } E^n \\
&= \text{space of oriented co-directions in real } E^n,
\end{aligned}
$$

a real form of G/P; cf. 4.7.

Since $G' = T^{-1}\, G\, T$, we can exhibit G_0'/P_0', as well as G_0/P_0, as a real form of the complex contact manifold G/P. $T\, G_0'\, P^{-1}$ is the real form of $G = T\, G'\, T^{-1}$ consisting of the elements fixed under the conjugation obtained by transporting the conjugation $g' \to \bar{g}'$ of G' to G, namely

$$
g \to T\,\overline{(T^{-1} g T)}\, T^{-1} = S^{-1} \bar{g} S
$$

where $S = \bar{T} T^{-1}$. In the case of n odd,

$$
S = \left[\begin{array}{cc|cc}
0 & 0 & 1_{l-2} & 0 \\
0 & 1_2 & 0 & 0 \\
\hline
1_{l-2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1_2
\end{array}\right];
$$

in the case of n even, S has an additional initial row and column with a 1 in their common first entry and 0s elsewhere. ${}^t S\, A\, S = A$ and $S^2 = 1_{n+3}$, so the complex conjugation $\xi \to S^{-1}\bar{\xi}$ preserves the quadric Ω^{n+1}. A point or line of Ω^{n+1} is fixed under this conjugation exactly if it is the image under T of a real point or line of Ψ^{n+1}. The latter constitute the orbit on Ω^{n+1} of $T\, G_0'\, T^{-1}$. The isotropy subgroup in $T\, G_0'\, T^{-1}$ of the line l_0 of Ω^{n+1} is $T\, G_0'\, T^{-1} \cap P = T\, P_0'\, T^{-1}$. Furthermore, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in 5.1 is stable under the conjugation $X \to S^{-1}\, \bar{X}\, S$ of $\mathfrak{g}$; in fact, for

$$
H = \operatorname{diag}\,(h_1, \ldots, h_l \mid -h_1, \ldots, -h_l)
$$

in $\mathfrak{h}$, we have

$$
S^{-1} \bar{H}\, S = \operatorname{diag}\,(-\bar{h}_1, \ldots, -\bar{h}_{l-2}, \bar{h}_{l-1}, \bar{h}_l \mid \bar{h}_1, \ldots, \bar{h}_{l-2}, -\bar{h}_{l-1}, -\bar{h}_l),
$$

in case of n odd; the maximal root $\rho = -\delta_{l-1} - \delta_l$ is real, $\overline{\rho(S^{-1} \bar{H} S)} = \rho(H)$. Hence, the contact structure on $TG_0' T^{-1}/TP_0' T^{-1}$ is that obtained from G/P by 2.11. We conclude:

G_0/P_0 and $TG_0' T^{-1}/TP_0' T^{-1}$, the latter isomorphic to G_0'/P_0', are two reals forms of the complex contact manifold G/P.

5.6. We observed in 5.3 that the space of co-directions in complex projective space P^3, by means of Plücker's line geometry, is isomorphic to the space of lines in the quadric Ω^4 in complex P^5, and that this isomorphism makes real line geometry correspond to a real form of Ω^4. We found in 5.4 and 5.5 that the space of oriented co-directions in complex Euclidean space E^3 of Lie's higher sphere geometry, which is the space of lines in the quadric Ψ^4 in complex P^5, is isomorphic to the space of lines in the quadric Ω^4 also, and that this isomorphism makes real sphere geometry correspond to a second real form of Ω^4. That is, real line geometry and real sphere geometry are two distinct real forms of complex line geometry. The line-sphere transformation establishes the isomorphism of the spaces of lines in Ψ^4 and lines in Ω^4. The former places real sphere geometry in the foreground, the latter, real line geometry.

5.7. The isomorphism of 5.3 may be used to describe sphere geometry in terms of co-directions in complex P^3. Real sphere geometry then leads to the real form $PSU(2,2)$ of $PSL(4; \mathbf{C})$.

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(Reçu le 20 mars 1978)

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