

# 3. CO-DIRECTIONS IN PROJECTIVE SPACE

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is one-to-one on an open neighborhood  $U$  of  $x_0$  in  $G/P$  and  $(\exp X) \cdot x_0$  is identifiable as the point  $(x_1, \dots, x_n)$  and the incident hyperplane

$$x'_n - x_n = p_1(x'_1 - x_1) + \dots + p_{n-1}(x'_{n-1} - x_{n-1}).$$

Now,  $(\exp X) \cdot x_0 \rightarrow (\exp X) \cdot b_0$  is a section of the bundle  $G/P_1$  over  $U$  and, via this section, the form  $\omega$  on  $G/P_1$  pulls down to

$$\omega_0((\exp X)^{-1} d(\exp X))$$

which, when expressed in terms of  $x_1, \dots, x_n, p_1, \dots, p_{n-1}$ , will be identified with

$$dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1}$$

up to a constant multiple  $a \neq 0$ . For this latter calculation we will use

$$\begin{aligned} (\exp X)^{-1} d(\exp X) &= \frac{1 - e^{-ad X}}{ad X} (dX) \\ &= dX - \frac{1}{2} [X, dX] + \frac{1}{6} [X, [X, dX]] - \dots \end{aligned}$$

[4, (10.2)], a series which is finite since  $\mathfrak{m}$  is nilpotent. In fact, our choice of  $X$  will make the series for  $\exp X$  themselves finite. The constant  $a \neq 0$  could be made unity by using instead the section  $(\exp X) \cdot x_0 \rightarrow (\exp X)g^{-1} \cdot b_0$ , where  $g$  in  $P$  is chosen so that  $\chi(g) = a$ . This amounts to following the original section by  $R_a^{-1}$  in the bundle.

### 3. CO-DIRECTIONS IN PROJECTIVE SPACE

The contact structure on the  $(2n-1)$ -dimensional space of co-directions in complex projective space  $P^n$ , described in 2.5, is obtained when the construction of 2.10 is carried out for the simple complex Lie algebra of type  $A_n, n \geq 1$ .

3.1 Let  $\mathfrak{g} = \mathfrak{sl}(n+1; \mathbf{C})$ , complex  $(n+1)$  by  $(n+1)$  matrices of trace zero. For Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  take the diagonal matrices of  $\mathfrak{g}$ . Let  $\delta_i, i = 0, 1, \dots, n$  be the linear function on  $\mathfrak{h}$  which assigns to  $H = \text{diag}(h_1, \dots, h_n)$  in  $\mathfrak{h}$  the  $i^{\text{th}}$  diagonal element:  $\delta_i(H) = h_i$ . The roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$  are

$$\begin{aligned} \delta_i - \delta_j \quad i, j = 0, 1, \dots, n \\ \text{and } i \neq j \end{aligned}$$

and the root vector  $E_\alpha$  corresponding to the root  $\alpha$  is

$$E_{\delta_i - \delta_j} = E_{ij},$$

the matrix with 1 in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column and 0s elsewhere [4, (16.2)]. A system of simple roots is

$$\delta_0 - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n,$$

for which the maximal root is

$$\rho = (\delta_0 - \delta_1) + (\delta_1 - \delta_2) + \dots + (\delta_{n-1} - \delta_n) = \delta_0 - \delta_n$$

[4, App., Table E]. The Killing form of  $\mathfrak{g}$  is  $\langle X, Y \rangle = 2(n+1) \text{tr}(XY)$ , but we replace this with  $\langle X, Y \rangle = \text{tr}(XY)$  for convenience. Then the  $H_\alpha$  in  $\mathfrak{h}$  are given by

$$H_{\delta_i - \delta_j} = \text{diag}(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$$

with 1 and  $-1$  in the  $i^{\text{th}}$  and  $j^{\text{th}}$  entry, respectively. Especially,

$$H_\rho = \text{diag}(1, 0, \dots, 0, -1).$$

We have

$$\langle H_\rho, H_{\delta_i - \delta_j} \rangle \begin{cases} < 0 & j = 0 \text{ or } i = n \\ \geq 0 & \text{otherwise} \end{cases},$$

so that  $\mathfrak{p}$  in (i) of 2.9 consists of matrices of the form

$$\begin{bmatrix} * & * & \cdots & * \\ & \vdots & & \vdots \\ 0 & & & \\ & * & \cdots & * \\ & & & \\ & & & \\ 0 & \cdots & 0 & * \end{bmatrix}$$

of trace zero, where the starred entries are arbitrary.

3.2 The connected centerless simple group  $G = PSL(n+1; \mathbf{C}) = SL(n+1; \mathbf{C})/\{\text{center}\}$  is transitive on the space consisting of points  $x$  and incident hyperplanes  $u, ux = 0$ , in  $P^n$ , as in 2.5. The isotropy subgroup  $P$  of the incident point and hyperplane

$$x_0 = {}^t(1, 0, \dots, 0), \quad u_0 = (0, \dots, 0, 1)$$

has exactly  $\mathfrak{p}$  for its Lie algebra. Hence, the homogeneous contact manifold which the construction of 2.10 gives is

$$\begin{aligned} G/P &= \text{space of incident points and hyperplanes in } P^n \\ &= \text{space of co-directions in complex } P^n. \end{aligned}$$

3.3 Let  $\mathfrak{m}$  be the  $(2n-1)$ -dimensional supplement to  $\mathfrak{p}$  in  $\mathfrak{g}$  consisting of matrices of the form

$$\begin{bmatrix} 0 & & & & & \\ & * & & & 0 & \\ & \vdots & & & & \\ & & & & & \\ & & & & * & 0 \end{bmatrix};$$

cf. 2.12. The product of any two matrices of  $\mathfrak{m}$  has a nonzero entry only in the  $n^{\text{th}}$  row and  $0^{\text{th}}$  column; the product of any three is zero. Set

$$X = \begin{bmatrix} 0 & & & & & \\ & x_1 & & & 0 & \\ & \vdots & & & & \\ & & & & & \\ & & & & x_{n-1} & \\ x_n - \frac{1}{2} \sum p_i x_i & p_1 \cdots p_{n-1} & & & & 0 \end{bmatrix},$$

where the summation is over  $i = 1, 2, \dots, n-1$ .  $X$  is in  $\mathfrak{m}$  and

$$\exp X = 1_{n+1} + X + \frac{1}{2} X^2 =$$

$$\begin{bmatrix} 1 & & & & & \\ & x_1 & & & 0 & \\ & \vdots & & & & \\ & & & & & \\ & & & & x_{n-1} & \\ & & & & & \\ x_n & p_1 \cdots p_{n-1} & & & & 1 \end{bmatrix},$$

$$(\exp X)^{-1} = 1_{n+1} - X + \frac{1}{2} X^2 = \begin{bmatrix} 1 & & & & & & & & & & \\ & -x_1 & & & & & & & & & 0 \\ & \vdots & & & & & & & & & \\ & & -x_{n-1} & & & & & & & & 0 \\ & & & -x_n + \sum p_i x_i & & & -p_1 & \dots & & & -p_{n-1} \\ & & & & & & & & & & 1 \end{bmatrix}.$$

The point

$$x = (\exp X) \cdot x_0 = {}^t(1, x_1, \dots, x_n)$$

is incident with the hyperplane

$$u = u_0 \cdot (\exp X)^{-1} = (-x_n + \sum p_i x_i, -p_1, \dots, -p_{n-1}, 1),$$

and the hyperplane  $ux' = 0$ ,  $x' = {}^t(1, x'_1, \dots, x'_n)$ , is

$$x'_n - x_n = p_1(x'_1 - x_1) + \dots + p_{n-1}(x'_{n-1} - x_{n-1}).$$

Thus, this choice of  $X$  establishes the classically identifiable coordinates  $x_1, \dots, x_n, p_1, \dots, p_{n-1}$  on  $G/P$ .

3.8 From  $\rho = \delta_0 - \delta_n$ , we have  $W = E_\rho = E_{0n}$  in (iii) of 2.9 and  $\omega_0(X) = \langle W, X \rangle$  is the  $n0$ -entry of  $X$ . The form  $\omega$  on  $G/P$  is obtained as  $\omega = \omega_0((\exp X)^{-1} d(\exp X))$  with

$$(\exp X)^{-1} d(\exp X) = dX - \frac{1}{2} [X, dX]$$

as in 2.12. For  $X$  as in 3.3, the only nonzero entry in  $[X, dX]$  is the  $n0^{th}$  and it is  $\sum p_i dx_i = -\sum x_i dp_i$ . Hence

$$(\exp X)^{-1} d(\exp X) = \begin{bmatrix} 0 & & & & & & & & & & \\ & dx_1 & & & & & & & & & 0 \\ & \vdots & & & & & & & & & \\ & & dx_{n-1} & & & & & & & & \\ & & & dx_n - \sum p_i dx_i & & & dp_1 & \dots & dp_{n-1} & & 0 \end{bmatrix},$$

and the  $n0$ -entry is

$$\omega = dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1}.$$

This identifies the contact structure with the classical one as in 2.12.

3.5 The real contact structure on the  $(2n-1)$ -dimensional space of co-directions in real projective space  $P^n$  is described by viewing all quantities in the foregoing discussion as being real. Especially,  $G_0$  of 2.11 is the connected centerless group  $PSL(n+1; \mathbf{R})$  consisting of real contact automorphisms.

#### 4. HIGHER SPHERE GEOMETRY

4.1 In complex Euclidean space  $E^n$ , the equation

$$x_1'^2 + \dots + x_n'^2 - 2a_1 x_1' - \dots - 2a_n x_n' + C = 0$$

describes a sphere with center  $(a_1, \dots, a_n)$  and complex radius  $r$  given by

$$r^2 = a_1^2 + \dots + a_n^2 - C.$$

When  $r \neq 0$ , the two choices of sign for  $r$  is said to give two "orientations" to the sphere. Thus, the  $n+2$  coordinates  $a_1, \dots, a_n, r, C$ , which are related by

$$a_1^2 + \dots + a_n^2 - r^2 - C = 0,$$

describe the space of oriented spheres in  $E^n$  [6, §25].

Introduce homogeneous coordinates by

$$a_i = \frac{\alpha_i}{v}, \quad r = \frac{\lambda}{v}, \quad C = \frac{\mu}{v},$$

$i = 1, 2, \dots, n$ . Then the oriented spheres of  $E^n$  correspond to certain points of the quadric  $\Psi^{n+1}$  in  $P^{n+2}$  described by

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu v = 0.$$

The sphere corresponding to the point  $(\alpha_1, \dots, \alpha_n, \lambda, \mu, v)$  of  $\Psi^{n+1}$  is

$$v(x_1'^2 + \dots + x_n'^2) - 2\alpha_1 x_1' - \dots - 2\alpha_n x_n' + \mu = 0.$$

Ordinary spheres have finite nonzero radius  $r$ , so  $v \neq 0$ . For  $v = 0$ , we obtain oriented hyperplanes. For  $\lambda = 0$ , we obtain point spheres or hyperplanes with isotropic hyperplane coordinate vector; these carry no