FIFTEEN CHARACTERIZATIONS OF RATIONAL DOUBLE POINTS AND SIMPLE CRITICAL POINTS

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by Alan H. DURFEE [1])

Rational double points of algebraic surfaces and simple critical points of complex analytic functions in several variables can be characterized in many ways, all of which involve some form of finiteness. These characterizations center on a list of polynomials (the simplest of which is $x^k + y^2 + z^2$), the Dynkin diagrams A_k, D_k and E_k, and the finite subgroups of the group of unit quaternions S^3 (Table 1).

This paper, which is expository in nature, is divided into two main sections. The first, Part A, consists of seven characterizations (numbered A1 through A7) of rational double points drawn from the work of algebraic geometers, among them Artin, Brieskorn, Du Val, Kirby and Laufer. A singularity of a complex analytic germ in $\mathbf{C}^3$ is a rational double point if a certain analytic cohomology group calculated from its resolution vanishes. It is then shown that the minimal resolution of this singularity must correspond to one of the Dynkin diagrams listed above and that the germ must be isomorphic to the zero locus of one of the germs listed in column 1 of Table 1. In terms of the method of resolution, these singularities are absolutely isolated double points. They are also quotient singularities and have finite local fundamental group. In addition, a limit involving volumes must be finite. The introduction to [Du Val 3] gives an historical account of the rational double points.

Part B contains nine characterizations (numbered B1 through B9) of simple critical points of complex analytic functions in several variables. These characterizations, the work of A'Campo, Arnold, Saito, Tjurina and others, involve the space of moduli of all germs, the quadratic form on the Milnor fiber, the monodromy group, the minimum number of critical values of a nearby Morse function, and the weights of weighted homogeneous polynomials. Parts A and B together present a total of

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fifteen characterizations, since Characterization A1 coincides with Characterization B2.

Most of the characterizations of Part B are shown to be equivalent to Characterization B1. Other links between the two sets of characterizations are provided by Theorem 12.2, which shows that Characterizations A2 and B5 are equivalent, and a recent result (Theorem 11.1) partially connecting Characterizations A2 and B3. Part B also contains a summary of pertinent work of Mather and Arnold.

There are two appendices. The first gives nine characterizations of simple elliptic singularities and almost-simple critical points. They are the next most reasonable class of singularities after rational double points, and can be characterized as being "infinite but not too infinite". All remaining singularities are "very infinite" in various senses. The second appendix contains Looijenga's proof that the monodromy group of the minimal hyperbolic germs has exponential growth.

This paper is an expanded version of a series of lectures given at the University of Maryland in the spring of 1976, and I thank the department of mathematics for its hospitality. The lectures were inspired by an unpublished talk given by E. Brieskorn at the American Mathematical Society Summer Institute in Algebraic Geometry in Arcata (1974). I also thank E. Looijenga and J. Wahl for helpful comments.

A. Seven characterizations of rational double points

Theorem A. *Let* $f(x, y, z)$ *be the germ at the origin* $\mathbf{0}$ *of a complex analytic function, and suppose that* $f(\mathbf{0}) = 0$ *and that the origin is an isolated critical point of* f. *Then characterizations A1 through A7 (which are listed below) are equivalent.*

1. Complex analytic spaces

Let V be the germ at $\mathbf{v}$ of a normal two-dimensional complex analytic space with a singularity at $\mathbf{v}$. (The definitions of these terms can be found in [Laufer 1].) For example, V could be $f^{-1}(0)$, where f is as in the hypotheses of Theorem A. Conversely, if V is embedded in $\mathbf{C}^3$ with $\mathbf{v}$ the origin, there is a germ f as above such that V is isomorphic to $f^{-1}(0)$ [Gunning and Rossi, p. 113]. The singularity is isolated since V is normal. Two such germs V and W embedded in $\mathbf{C}^n$ at the origin are *isomorphic* if there is a germ of an analytic automorphism of $\mathbf{C}^n$ fixing the origin and taking V to W.

Characterization A1. The analytic set $f^{-1}(0)$ is isomorphic to the zero locus of one of the functions listed in column 1 of Table 1.

2. Rational singularities

A *resolution* of a germ of a normal surface singularity V as above is a complex analytic manifold M and an analytic map $\pi: M \to V$ that is surjective and proper (compact fibers) such that its restriction to $M - \pi^{-1}(\mathbf{v})$ is an analytic isomorphism, and $M - \pi^{-1}(\mathbf{v})$ is dense in M. Resolutions exist, and can be computed with a certain amount of effort. The article [Lipman 2] contains a general discussion of resolutions, and [Laufer 1] and [Hirzebruch, Neumann, and Koh, §9] give a detailed method with examples.

Among all resolutions there is a *minimal resolution* $\pi: M \to V$ that has the following universal mapping property: Given any other resolution $\pi': M' \to V$, there is a unique map $\rho: M' \to M$ with $\pi' = \pi \circ \rho$.

The *geometric genus* p of V is the dimension of the complex vector space $H^1(M, \mathcal{O}_M)$, where M is any resolution of V, and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M [Artin; Wagreich 1, §1.4; Brieskorn 2; Laufer 2]. (V is assumed Stein.) This number is finite, and independent of the choice of resolution. It may alternately be defined as the dimension of the stalk at the origin of the sheaf $R^1 \pi_* \mathcal{O}_M$ on V. The idea behind this definition is that M is a collection of "thickened" curves, and that the genus of a curve X is the dimension of $H^1(X, \mathcal{O}_X)$. For example, $H^1(M, \mathcal{O}_M) = 0$ if M is the total space of a line bundle over a curve of genus zero. On the other hand, dim $H^1(M, \mathcal{O}_M) = k(k-1)(k-2)/6$ if M is a line bundle of Chern class $-k$ over a curve of genus $(k-1)(k-2)/2$ (the minimal resolution of $f(x, y, z) = x^k + y^k + z^k$). In terms of V alone, p is the dimension of the space of holomorphic two-forms on $V - \mathbf{v}$ divided by square-integrable forms [Laufer 2, Theorem 3.4]. Another formula for p in terms of topological invariants of the resolution M and the nearby fiber F (see §11) is given in [Laufer 6].

The analytic set V has a *rational* singularity if $p = 0$. A rational singularity embeds in codimension 1 if and only if it is a double point (its local ring is of multiplicity two) [Artin, Corollary 6].

Characterization A2. The singularity of $f^{-1}(0)$ is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

3. Exceptional sets

Let V be as above, and let $\pi: M \to V$ be a resolution of V. The *exceptional set* $E = \pi^{-1}(\mathbf{v})$ is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves $E_1, \ldots, E_s$. It is possible to arrange that the E_i are non-singular, the intersection of E_i and E_j is transverse for $i \neq j$, and no three E_i meet at a point. Such a resolution is called *good.* If, in addition, the intersection of E_i and E_j is empty or one point, the resolution is *very good*; this is possible to arrange as well.

Suppose that the resolution is good. Let $E_i \cdot E_j$ equal the number of points of intersection of E_i and E_j if $i \neq j$ (always a non-negative integer), or the first Chern class of the normal bundle to E_i evaluated on the orientation class of E_i if $i = j$ (the self-intersection of E_i). The matrix $\{E_i \cdot E_j\}$ is called the *intersection matrix of the resolution.* It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves $E = E_1 \cup \ldots \cup E_s$ in a two-dimensional manifold M with negative definite intersection matrix $\{E_i \cdot E_j\}$, a theorem of Grauert says that the quotient space M/E has a normal complex structure and that the projection map $M \to M/E$ is analytic [Laufer 1, p. 60].

Characterization A3. The minimal resolution of $f^{-1}(0)$ is very good, and its exceptional set consists of curves of genus 0 and self-intersection -2.

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

(i) Let $M \to V$ be a resolution of a normal singularity V as above. There is a certain unique non-zero divisor $Z = \Sigma n_i E_i$ on M with $n_i \geqslant 0$ called the *fundamental cycle,* and it is shown that the singularity of V is rational if and only if the analytic Euler characteristic $\chi(Z)$ of Z is 1 (that is, the arithmetic genus of Z is 0) [Artin, Theorem 3]. It is easy to see that the support of Z is the whole exceptional set of E.

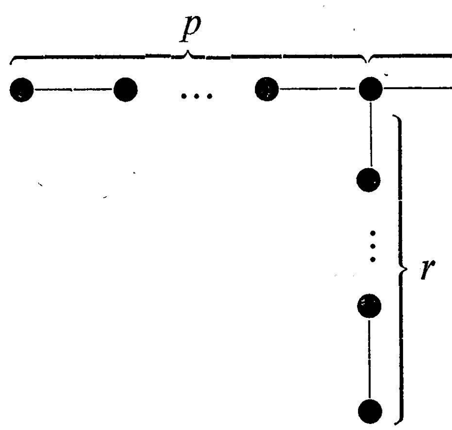
(ii) Any resolution of a rational singularity V is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].

(iii) A rational singularity V embeds in codimension one if and only if it is a double point, which is true if and only if $Z^2 = -2$ [Artin, Corollary 6].

(A2) ⇒ (A3): We only need show $E_i^2 = -2$ for all i. Certainly $E_i^2 \leqslant -2$, since if $E_i^2 = -1$ the resolution could be contracted by Castelnuovo's criterion, and $E_i^2 \geqslant 0$ would contradict the fact that the matrix $\{E_i \cdot E_j\}$ is negative definite. Let K be the canonical class of M. (This exists since V is Gorenstein; see for instance [Durfee 2].) The adjunction formula $-E_i \cdot K = E_i^2 + 2$ then shows that $E_i \cdot K \geqslant 0$ for each i. The Riemann-Roch Theorem $\chi(Z) = -\frac{1}{2}(Z^2 + Z \cdot K)$ implies that $Z \cdot K = 0$. Thus $0 = Z \cdot K \geqslant (E_1 + \ldots + E_s) \cdot K \geqslant E_i \cdot K \geqslant 0$. Hence $E_i \cdot K = 0$ for all i, so again by the adjunction formula, $E_i^2 = -2$.

(A3) ⇒ (A2): The adjunction formula implies that $E_i \cdot K = 0$ for all i; since the matrix $\{E_i \cdot E_j\}$ is negative definite, $K = 0$. Thus $\chi(Z) = \frac{1}{2} Z^2$ by the Riemann-Roch Theorem. Since $\chi(Z) \leqslant 1$ and $Z^2 < 0$ (again since $\{E_i \cdot E_j\}$ is negative definite), $\chi(Z)$ must be 1 and Z^2 must be -2. This completes the proof.

Now, exactly what exceptional sets satisfy Characterization A3? First some algebra. It is possible to associate a weighted graph to any symmetric integral bilinear form $\langle \, , \rangle$ on a free module with basis $e_1, \ldots, e_s$ satisfying $\langle e_i, e_j \rangle \geqslant 0$ for $i \neq j$: The vertices of the graph are $v_1, \ldots, v_s$, two vertices v_i and v_j are joined by $\langle e_i, e_j \rangle$ edges, and the vertex v_i is weighted by the integer $\langle e_i, e_i \rangle$. Conversely, a weighted graph defines such a bilinear form. Let $T_{p,q,r}$ be the weighted graph

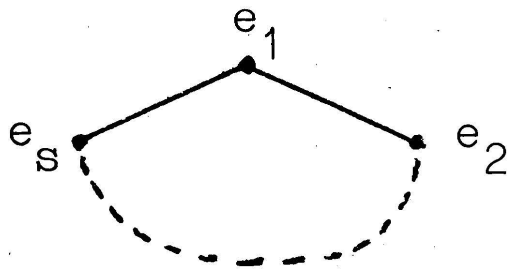


where p, q, and r are positive integers, and all vertices are weighted by -2.

Lemma 3.1 [Hirzebruch 2, p. 217]. The only connected graphs weighted by -2 and whose associated bilinear form is negative definite are of type $T_{p,q,r}$, where p, q, and r are positive integers satisfying $p^{-1}+q^{-1}+r^{-1} > 1$.

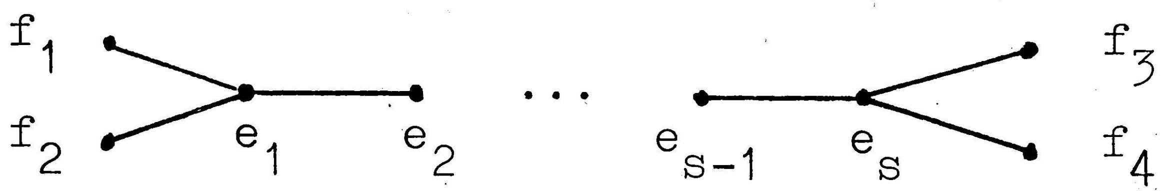
Proof. (a) If the bilinear form associated to a graph is negative definite, so is the bilinear form associated to any subgraph.

(b) The graph ($s \geqslant 2$)



where all vertices $e_1, \ldots, e_s$ are weighted by -2, is not negative definite, since $(e_1 + \ldots + e_s)^2 = 0$.

(c) The graph



where all vertices are weighted by -2, is not negative definite, since $(2e_1 + \ldots + 2e_s + f_1 + \ldots + f_4)^2 = 0$.

Thus the graph must be of the form $T_{p,q,r}$. An elementary argument shows that the bilinear form of $T_{p,q,r}$ is isomorphic over the rationals to the direct sum of a negative definite form and the one-dimensional form $\langle 1 - p^{-1} - q^{-1} - r^{-1} \rangle$. Hence $T_{p,q,r}$ is negative definite if and only if $p^{-1} + q^{-1} + r^{-1} > 1$. This proves the lemma.

The only triples of positive integers (p, q, r) satisfying $p^{-1} + q^{-1} + r^{-1} > 1$ are of course just $(1, 1, r)$ for $r \geqslant 1$, $(2, 2, r)$ for $r \geqslant 2$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$.

The *dual graph of a resolution of a singularity* is defined to be the weighted graph associated to the intersection matrix of the resolution. Applying the above facts, we see that Characterization A3 is equivalent to:

Characterization A3′. The minimal resolution of $f^{-1}(0)$ is listed in column (3) of Table 1.

Next we show that Characterization A1 and A3 are equivalent. Characterization A1 implies Characterization A3 since the singularities of the functions f listed in column 1 of Table 1 have minimal resolutions as in column 3. (I believe that this first appeared in [Hirzebruch 1].) The converse follows since the singularities listed are taut [Brieskorn 2; Tjurina 3; Laufer 4]. (Two resolutions $\pi: M \to V$ and $\pi': M' \to V'$ are *topologically equivalent* if their exceptional sets are homeomorphic by a homeomorphism preserving the self-intersection numbers. A singularity V is *taut* if any other singularity with a good resolution topologically equivalent to a good resolution of V is then isomorphic to V.)

The classification of rational double points has been generalized in several ways: to rational triple points [Artin, p. 135], to elliptic singularities [Wagreich 1], and to minimally elliptic singularities [Laufer 5]. The Dynkin diagrams B_n, C_n, F_4 and G_2 occur when resolving singularities over non-algebraically closed fields [Lipman 1]. There is also a relation with simple complex Lie groups [Brieskorn 3].

4. Absolutely isolated double points

There are at least three methods of resolving the singularity of the germ of a normal two-dimensional complex space V. The first method is one of local uniformization; this is originally due to Jung, and is described in detail in [Laufer 1]. The second method, due to Zariski, is to alternately blow up points and normalize. The third method (which generalizes to higher dimensions), is to blow up points and non-singular curves.

The singularity of V is *absolutely isolated* if it may be resolved by blowing up points alone, that is, it is not necessary to normalize or blow up curves. For example, the singularity of the zero locus of $f(x, y, z) = x^k + y^k + z^k$ is absolutely isolated, since it may be resolved by blowing up the origin once.

The singularity of V is a *double point* if its local ring is of multiplicity two. If V is $f^{-1}(0)$, this is equivalent to the lowest non-zero homogeneous term in the power series expansion of f being quadratic.

Characterization A4. The singularity of $f^{-1}(0)$ is an absolutely isolated double point.

The equivalence of Characterizations A1 and A4 was proved directly in [Kirby]. Later, it was shown [Tjurina 2; Lipman 1] that all rational singularities are absolutely isolated (thus showing Characterization A2 implies A4), and in [Brieskorn 1, Satz 1] that A4 implies A3.

5. Quotient singularities

Let U be a neighborhood of the origin $\mathbf{0}$ in $\mathbf{C}^2$ and let H be a finite group of analytic automorphisms of U fixing $\mathbf{0}$. The quotient space U/H has the structure of a normal two-dimensional complex analytic space with an isolated singularity, and the projection map $U \to U/H$ is analytic [Cartan]. An analytic space V is called a *quotient singularity* if there is a U and H as above such that V is isomorphic to U/H.

An important example of a quotient singularity is $\mathbf{C}^2/G$, where G is some finite subgroup of $GL(2, \mathbf{C})$. The space $\mathbf{C}^2/G$ is not just analytic, but algebraic. For any finite subgroup G of $GL(2, \mathbf{C})$, the ring of functions on the algebraic variety $\mathbf{C}^2/G$ is isomorphic to the subring of invariant polynomials in $GL(2, \mathbf{C})$. Hence to find $\mathbf{C}^2/G$ it suffices to find this subring of invariant polynomials. Note that a finite subgroup G of $GL(2, \mathbf{C})$ or $SL(2, \mathbf{C})$ is conjugate to a finite subgroup of $U(2)$ or $SU(2)$ respectively, since it is possible to choose an invariant Hermitian metric on $\mathbf{C}^2$. A subgroup $G \subset GL(2, \mathbf{C})$ is *small* if no $g \in G$ has 1 as an eigenvalue of multiplicity one. [Prill, p. 380].

Proposition 5.1. *Let V be the germ of a normal two-dimensional complex analytic space. The following statements are equivalent.*

(a) *V is a quotient singularity.*

(b) *V is isomorphic to $\mathbf{C}^2/G$, for some finite subgroup G of* $GL(2, \mathbf{C})$.

(c) *V is isomorphic to $\mathbf{C}^2/G$, for some small finite subgroup of* $GL(2, \mathbf{C})$.

Condition (a) implies condition (b) by the usual linearization argument [Brieskorn 2, Lemma 2.2]. It is shown in [Prill, p. 380] that condition (b) implies condition (c). Obviously (c) implies (a). The following theorem is also proved in [Prill]: Let G and G' be small finite subgroups of $GL(2, \mathbf{C})$. Then the analytic spaces $\mathbf{C}^2/G$ and $\mathbf{C}^2/G'$ are isomorphic if and only if G and G' are conjugate.

Characterization A5. The analytic space $f^{-1}(0)$ is a quotient singularity.

Since quotient singularities are rational [Brieskorn 2, p. 340], Characterization A5 implies Characterization A2. The converse will follow in round-about fashion.

Consider $SU(2)$, which is of course isomorphic to the group S^3 of unit quaternions. The finite subgroups of S^3 are the cyclic group and the inverse images of the finite subgroups of the rotation group $SO(3)$ under the double cover $S^3 \to SO(3)$; these groups are listed in column 5 of Table 1.

PROPOSITION 5.2. *Let* G *be a non-trivial finite subgroup of* $SU(2)$ *as listed in column 5 of Table* 1. *Then* $\mathbf{C}^2/G$ *is isomorphic to* $f^{-1}(0)$, *where* f *is the corresponding polynomial in column 1.*

In particular, for each polynomial f in column 1 of Table 1 the analytic space $f^{-1}(0)$ is isomorphic to a quotient singularity. This proposition is proved by classical invariant theory. For the cyclic group it is easy: Let $G \subset SU(2)$ be the cyclic group of order k, generated by the transformation $(u, v) \to (\eta u, \eta^{-1} v)$ where η is a primitive k-th root of unity. Then we claim that $\mathbf{C}^2/G$ is isomorphic to

$$V = \{(x, y, z) \in \mathbf{C}^3 : x^k = yz\}.$$

Let $p_1(u, v) = uv$, $p_2(u, v) = u^k$, $p_3(u, v) = v^k$, and let $p = (p_1, p_2, p_3)$ define a map of $\mathbf{C}^2$ to $\mathbf{C}^3$. The image of p is exactly V. Since $p_i(gu, gv) = p_i(u, v)$ for all g in G, the map p induces a map $\bar{p}$ of $\mathbf{C}^2/G$ to V. The map $\bar{p}$ is easily seen to be injective, and thus is an isomorphism, since $\mathbf{C}^2/G$ and V are normal.

The proof for the other finite subgroups G of S^3 is similar, and may be found in [Du Val 3]: The elements of G are listed, the subring R of $\mathbf{C}[u, v]$ of invariant polynomials is found to be generated by three homogeneous polynomials p_1, p_2, p_3 of various degrees, and they satisfy exactly one weighted homogeneous relation $f(p_1, p_2, p_3) = 0$. It follows that $\mathbf{C}^2/G$ is isomorphic to the zero locus of f. Special cases of this proof go back to [Klein]. It is also possible to give a simpler uniform proof using vertices, edges, and faces when G is the commutator subgroup $[H, H]$ of another finite subgroup H of S^3 [Milnor 2, §4].

[Du Val 3, §30] gives a geometric description of the links of these singularities as regular solids with opposite faces identified. (The *link* of a germ $V \subset \mathbf{C}^n$ at $\mathbf{v}$ is V intersected with a suitably small sphere about $\mathbf{v}$.)

The finite subgroups of $GL(2, \mathbf{C})$ are listed in [Du Val 3, §21] and the corresponding quotient singularities are studied in [Brieskorn 2, p. 348]. The ring of invariant polynomials has been computed for the cyclic and dihedral subgroups [Riemenschneider 1,2]. Generalizations of quotient singularities and their relation to weighted homogeneous polynomials may be found in [Milnor 2; Dolgachev].

Characterization A5′. The analytic space $f^{-1}(0)$ is isomorphic to $\mathbf{C}^2/G$, where G is a finite subgroup of $SU(2)$.

Proposition 5.2 shows that characterizations A5′ and A1 are equivalent. Clearly Characterization A5′ implies A5; since A5 implies A2, they are all equivalent.

COROLLARY 5.3. *Let* G *be a small finite subgroup of* $GL(2, \mathbf{C})$. *Then* $G \subset SL(2, \mathbf{C})$ *if and only if* $\mathbf{C}^2/G$ *embeds in codimension one.*

This corollary follows from the above case-by-case analysis. J. Wahl points out that it is also possible to prove it directly, using the following two facts:

Fact 1. Let G be a small finite subgroup of $GL(2, \mathbf{C})$. Then $G \subset SL(2, \mathbf{C})$ if and only if the singularity of $\mathbf{C}^2/G$ is Gorenstein.

This is a special case of [Watanabe]. A germ of a normal two-dimensional complex space is *Gorenstein* if there is a nowhere-vanishing holomorphic two-form on its regular points.

Fact 2. Let V be the germ at $\mathbf{v}$ of a two-dimensional rational singularity. Then V is Gorenstein if and only if V embeds in codimension 1.

Proof. Any singularity embedded in codimension one is Gorenstein. Conversely, suppose V is Gorenstein. Let $\pi: M \to V$ be the minimal resolution of V, and let $E_1 \cup \ldots \cup E_s = \pi^{-1}(v)$ be its exceptional set as in Section 3. Since V is Gorenstein, there is a divisor K on M (the *canonical class*) satisfying the adjunction formula. Furthermore $K \cdot E_i \geqslant 0$ for all i since the resolution is minimal, so $K \leqslant 0$ [Artin, bottom of p. 130]. If $K < 0$, then $-K > 0$, so arithmetic genus p of $-K$ satisfies $p(-K) \leqslant 0$ [Artin, Proposition 1]. On the other hand, $p(-K) = 1 - \chi(-K) = 1$ by the Riemann-Roch Theorem, a contradiction. Hence $K = 0$. Thus $K \cdot E_i = 0$ for all i, so V is a double point and embeds in codimension one, as in the proof that Characterization A3 implies Characterization A2.

6. THE LOCAL FUNDAMENTAL GROUP

Let V be the germ of a normal two-dimensional complex analytic space with an isolated singularity at $\mathbf{v}$. Without loss of generality, we may assume that V is a *good neighborhood* of $\mathbf{v}$, that is, that there is a neighborhood basis V_i of $\mathbf{v}$ in V such that each $V_i - \mathbf{v}$ is a deformation retract of $V - \mathbf{v}$ [Prill]. The *local fundamental group* of V at $\mathbf{v}$ is then defined as $\pi_1(V - \mathbf{v})$. This group is trivial if and only if V is nonsingular at $\mathbf{v}$ [Mumford].

PROPOSITION 5.1 (bis). *The following statement is equivalent to those listed above.*

(d) *The local fundamental group of V is finite.*

It is shown in [Prill, p. 381; Brieskorn 2, p. 344] that conditions (a) and (d) are equivalent.

Characterization A6. The local fundamental group of $f^{-1}(0)$ is finite.

Thus Characterizations A5 and A6 are equivalent.

There is an algorithm for computing the local fundamental group of V from a resolution [Mumford], and singularities V with finite, nilpotent and solvable local fundamental group have been classified [Brieskorn 2; Wagreich 2]. When V is a complete intersection, this classification is particularly simple [Durfee 2, Proposition 3.3].

7. VOLUME

Let $f(x, y, z)$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose that $f(\mathbf{0}) = 0$ and that the origin is an isolated critical point of f. There is an $\varepsilon > 0$ such that $f^{-1}(\mathbf{0})$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leqslant \varepsilon$. (See Section 12.) For $t \in \mathbf{C}$, let

$$V_t = f^{-1}(t) \cap D_\varepsilon^6$$

where D_ε^6 is the closed disk of radius ε about $\mathbf{0}$. The function $f(x, y, z)$ takes the constant value t on V_t, so $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \equiv 0$ there. Hence a nowhere-vanishing holomorphic two-form ω_t on V_t may be defined by the equivalent expressions

$$\omega_t = \frac{dy \wedge dz}{\partial f / \partial x} = \frac{dz \wedge dx}{\partial f / \partial y} = \frac{dx \wedge dy}{\partial f / \partial z},$$

Characterization A7. The integral $\int_{V_0} \omega_0 \wedge \bar{\omega}_0$ is finite.

Note that the form $\omega_0 \wedge \bar{\omega}_0$ takes positive real values. The equivalence of Characterizations A2 and A7 is due to Laufer, and follows easily from his expression for the geometric genus in terms of forms [Laufer 2, Corollary 3.6].

A different formulation of this characterization is due to E. Looijenga (unpublished): Let $\Delta(r) = \{t \in \mathbf{C}: t < r\}$, let

$$X(r) = f^{-1}(\Delta(r)) \cap D_\varepsilon^6$$

and let vol $(X(r))$ be its volume in $\mathbf{C}^3$.

Characterization A7'. $\lim_{r \to 0} r^{-2} \operatorname{vol}(X(r))$ is finite.

Let $\omega = dx \wedge dy \wedge dz$, and note that $\omega \wedge \bar{\omega}$ is $8/i$ times the volume form of $\mathbf{C}^3$. Characterizations A7 and A7' are equivalent since

$$\lim_{r \to 0} \frac{1}{r^2} \operatorname{vol}(X(r)) = \lim_{r \to 0} \frac{i}{8r^2} \int_{X_r} \omega \wedge \bar{\omega}$$

$$= \lim_{r \to 0} \frac{i}{8r^2} \int_{\Delta(r)} \left(\int_{V_t} \omega_t \wedge \bar{\omega}_t \right) dt \wedge \overline{dt} \,,$$

but since

$$\int_{\Delta(r)} \left(\frac{i}{2}\right) dt \wedge \overline{dt} = \operatorname{vol}(\Delta(r)) = 2\pi r^2 \,,$$

the above limit equals

$$\frac{\pi}{2} \int_{V_0} \omega_0 \wedge \bar{\omega}_0 \,.$$

B. Nine characterizations of simple critical points

We switch our attention from the analytic set defined by the zero locus of an analytic function $f(x, y, z)$ to the function itself and the nature of its critical point. We also generalize to functions $f(z_0, \ldots, z_n)$ of an arbitrary number of variables. The characterizations in the following theorem will start in Section 9.

Theorem B. *Let $f(z_0, \ldots, z_n)$ with $n \geqslant 1$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that $\mathbf{0}$ is an isolated critical point of f. Then Characterizations B1 through B9 are equivalent.*

8. The classification of right equivalence classes

Let $\mathcal{O}$ be the set of germs f at the origin $\mathbf{0}$ of complex analytic functions on $\mathbf{C}^{n+1}$. (In other words, $\mathcal{O}$ is just the ring $\mathbf{C}\{z_0, \ldots, z_n\}$ of convergent power series.) The ring $\mathcal{O}$ is local with maximal ideal

$$\mathfrak{m} = \{f \in \mathcal{O} : f(\mathbf{0}) = 0\} \,.$$

Let

$$\Delta f = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

be the ideal in $\mathcal{O}$ generated by the partial derivatives of f.

Lemma 8.1. A germ f in $\mathfrak{m}$ has an isolated critical point at $\mathbf{0}$ if and only if there is a k such that $\mathfrak{m}^k \subset \Delta f \subset \mathfrak{m}$.

Proof. The germ f has a critical point at $\mathbf{0}$ if and only if $f \in \mathfrak{m}^2$, or equivalently, $\Delta f \subset \mathfrak{m}$. If this critical point is isolated, then the origin is an isolated zero of the functions $\partial f/\partial z_0, \ldots, \partial f/\partial z_n$. This is equivalent to saying that the set of common zeros of all the functions in the ideal Δf equals the set of common zeros of the ideal $\mathfrak{m}$. By the Nullstellensatz, there exist integers $l_0, \ldots, l_n$ such that $z_i^{l_i} \in \Delta f$. Setting $k = (n+1) \max \{l_0, \ldots, l_n\}$ gives $\mathfrak{m}^k \subset \Delta f$. Conversely, if $\mathfrak{m}^k \subset \Delta f$ then the origin is an isolated critical point. This proves the lemma.

Let $\mathcal{F}$ be the set of all germs in $\mathcal{O}$ vanishing at the origin and with an isolated critical point there. (This is the set of *finitely-determined* germs.) The *Milnor number* of a germ $f \in \mathcal{F}$ is

$$\mu = \dim_{\mathbf{C}} \mathcal{O}/\Delta f .$$

For a comprehensive discussion of μ, see [Orlik 2]. There are many ways to compute this number, aside from the above formula [Milnor 1, §§7, 10; A'Campo 1; Laufer 6]. The *(right) codimension* of f is $\mu - 1$.

Two germs f and g in $\mathcal{O}$ are *right equivalent* (written $f \sim g$) if there is a germ h of a complex analytic automorphism of $\mathbf{C}^{n+1}$ fixing $\mathbf{0}$ with $f \circ h = g$. The germs f and g are *contact equivalent* if there is an h as above such that the ideal generated by $f \circ h$ in $\mathcal{O}$ is equal to the ideal generated by g. This is equivalent to saying that the analytic sets $f^{-1}(0)$ and $g^{-1}(0)$ are isomorphic. Note that right-equivalent germs are contact equivalent.

Mather, Arnold, and others have classified germs of low Milnor number under right equivalence. The implicit function theorem shows, for example, that if $f(\mathbf{0}) = 0$ but the derivative of f does not vanish at $\mathbf{0}$, then f is right equivalent to the projection $(z_0, \ldots, z_n) \mapsto z_0$. If $f(\mathbf{0}) = 0$ and f has a non-degenerate critical point at $\mathbf{0}$, then $f(z_0, \ldots, z_n) \sim z_0^2 + \ldots + z_n^2$ by the Morse lemma.

Recall that the *k-jet* of a germ f in $\mathcal{O}$ is its power series expansion up to degree k. A germ $f \in \mathcal{O}$ is *k-determined* if any germ with the same k-jet as f is right equivalent to f. In particular, f is right equivalent to its own k-jet. A germ is *finitely determined* if it is k-determined for some $k < \infty$.

The fundamental lemmas used in the classification are as follows:

Lemma 8.2. If $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$ then f is k-determined.

For the proof, see [Arnold 1, Lemma 3.2; Zeeman, Theorem 2.9; Siersma, p. 8]. Note that $\mathfrak{m}^{k-1} \subset \Delta f$ implies that $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$. The *corank* of f is defined as $n + 1$ minus the rank of the Hessian matrix $\{(\partial^2 f / \partial z_i \partial z_j)(\mathbf{0})\}$. The proof of part (a) of the following lemma may be found in [Arnold 1, Lemma 4.1; Siersma Lemma 3.2].

Splitting Lemma 8.3. (a) Let $f(z_0, ..., z_n) \in \mathscr{F}$ be of corank $r + 1$. Then there is a $g(z_0, ..., z_r) \in \mathfrak{m}^3$ such that

$$f(z_0, \ldots, z_n) \sim g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 .$$

(b) Let $g(z_0, ..., z_r)$ and $g'(z_0, ..., z_r) \in \mathscr{F} \cap \mathfrak{m}^3$. If

$$g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 \sim g'(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2$$

then

$$g(z_0, \ldots, z_r) \sim g'(z_0, \ldots, z_r) .$$

The classification proceeds by low corank and low Milnor number. A germ of corank 0 is right equivalent to $z_0^2 + \ldots + z_n^2$, a germ of corank 1 and Milnor number $k > 1$ is right equivalent to $z_0^{k+1} + z_1^2 + \ldots + z_n^2$, and so forth. The proofs are not hard [Arnold 1, Zeeman, Siersma]. Table 2, for instance, includes all right-equivalence classes of germs with Milnor number $\mu \leqslant 9$.

9. Characterizations under right and contact equivalence

Characterization B1. The germ f is right equivalent to one of the germs in Table 2a.

Characterization B2. The germ f is contact equivalent to one of the germs in Table 2a.

When $n = 2$, Characterization B2 is the same as Characterization A1. Clearly Characterization B1 implies Characterization B2. Since all the germs in Table 2a are weighted homogeneous (§16), the converse follows from the next lemma.

Lemma 9.1. Let g be a weighted homogeneous polynomial, and suppose that a germ $f \in \mathscr{F}$ is contact equivalent to g. Then f is right equivalent to g.

Proof. To say that f is contact equivalent to g means that there is a germ of an analytic isomorphism $h: (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}^{n+1}, \mathbf{0})$ and a function $u: \mathbf{C}^{n+1} \to \mathbf{C}$ with $u(\mathbf{0}) \neq 0$ such that $f = u \cdot (g \circ h)$. Let $h = (h^0, ..., h^n)$ be the components of h, and suppose that g is weighted homogeneous with weights $(w_0, ..., w_n)$. Then,

$$\begin{aligned} f(z_0, ..., z_n) &= u(z_0, ..., z_n) \cdot g\left(h^0(z_0, ..., z_n), ..., h^n(z_0, ..., z_n)\right) \\ &= g\left((u(z_0, ..., z_n))^{1/w_0}\ h^0(z_0, ..., z_n), ...,\right. \\ &\qquad \left.(u(z_0, ..., z_n))^{1/w_n}\ h^n(z_0, ..., z_n)\right). \end{aligned}$$

Hence f is right equivalent to g.

10. Degeneration

Let J_k be the set of k-jets of germs in $\mathcal{O}$. There is a projection of $\mathcal{O}$ to J_k by mapping germs to their power series expansion truncated in degree k. The ring $\mathcal{O}$ becomes a topological space by letting a basis of open sets be inverse images of open sets in J_k, for all k.

The group of germs of analytic automorphisms fixing $\mathbf{0}$ acts on $\mathcal{O}$, and the orbits of this action *(right-equivalence orbits)* are the right-equivalence classes. Similarly, there is a contact equivalence group which acts on $\mathcal{O}$, and the orbits of this action *(contact-equivalence orbits)* are the contact equivalence classes [Mather, §2]. A right-equivalence orbit is always contained in a contact-equivalence orbit; Lemma 9.1 says that the right-equivalence orbit of a germ f in Table 2a or b equals its contact-equivalence orbit.

A subset A of $\mathcal{O}$ is said to *right* (or *contact*) *degenerate* to a subset B of $\mathcal{O}$ if the closure of the right (or contact) equivalence orbit of A contains B. If A degenerates to B, then B *simplifies* to A (written $A \leftarrow B$). Right degeneracy is also called *adjacency*. For example, when $n = 0$, the germ z_0^k right degenerates to the germ z_0^l for $k < l$, since the one-parameter family $tz_0^k + (1-t)\, z_0^l$ is z_0^l when $t = 0$, and is right-equivalent to z_0^k when $t \neq 0$. All germs of low codimension can be arranged according to right degeneracy in fascinating tables [Arnold 3; Siersma]. Table 3 lists some (but not all) of the simplifications that occur. The following proposition is a principal consequence of the work on degeneration.

Proposition 10.1.

(i) *The germs in Table 2a right simplify only to each other.*

(ii) *The germs in Table 2b right simplify only to the germs in Table 2a.*

(iii) *The germs in Table 2c right simplify only to the germs in Table 2b and 2a.*

(iv) *A germ in $\mathscr{F}$ not right equivalent to a germ in Table 2a, b, or c right simplifies to a germ in Table 2c.*

11. Simple germs and moduli

A germ $f \in \mathfrak{m}$ is said to be *right* (or *contact*) *simple* if there is a neighborhood of f in $\mathfrak{m}$ intersecting only finitely many right (or contact) equivalence orbits. In the language of algebraic geometry, a germ f is contact simple if and only if the versal deformation of $f^{-1}(0)$ contains only finitely many isomorphism classes of analytic spaces.

The germs in Table 2a are right and contact simple by Proposition 10.1. The germs in Table 2b are not contact simple (and hence not right simple): $\tilde{E}_6$ is a family of cones over non-singular elliptic curves in $\mathbf{C}P^2$, $\tilde{E}_7$ is a family of four lines through the origin in $\mathbf{C}^2$, and $\tilde{E}_8$ is a family of three parabolas [Arnold 1; Siersma]. Note that the germs of Table 2c form one-dimensional families under right equivalence, but all members of the family are contact equivalent [Laufer 4; Siersma p. 54]. Clearly if a germ g right simplifies to f and f is not right simple, then g is not right simple; the same applies to contact equivalence.

Characterization B3. The germ f is right simple.

Characterization B4. The germ f is contact simple.

The equivalence of Characterizations B1 and B3 follows from Proposition 10.1 and the above remarks [Arnold 1]. Characterization B3 implies Characterization B4 by definition. Conversely, a contact simple germ f which is not right simple right simplifies to a germ in Table 2b (by Proposition 10.1), but these are not contact simple. Hence f must be right simple.

The classification of simple germs has recently been extended to complete intersections [Giusti]. The *modality* of a germ f is defined in [Arnold 3]. A right-simple germ is zero-modal; all right equivalence classes of 1 and 2-modal germs have been listed [Arnold 2, 3, 5]. Moduli of resolutions of two-dimensional normal singularities are studied in [Laufer 3, 4]. The following result provides a connection between Characterizations A2 and B3.

Theorem 11.1 [Randell]. *For almost all germs $f(x, y, z)$ (in the sense of the Newton diagram), the geometric genus p of $f^{-1}(0)$ is less than or equal to the modality of f.*

12. The quadratic form

Let $f(z_0, ..., z_n)$ be a germ with $f(\mathbf{0}) = 0$ and an isolated critical point at $\mathbf{0}$ (that is, a germ in $\mathscr{F}$). There is an $\varepsilon > 0$ such that $f^{-1}(0)$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leqslant \varepsilon$. For suitably small $\delta > 0$, $f^{-1}(\delta')$ intersects the closed disk D_ε^{2n+2} of radius ε transversally for all $|\delta'| \leqslant \delta$. Let

$$F = f^{-1}(\delta) \cap D_\varepsilon^{2n+2}$$

be the *Milnor fiber* of f [Milnor 1]. The set F is a smooth real $2n$-manifold with boundary whose diffeomorphism type is independent of the choice of ε and δ. Furthermore, F is $(n-1)$-connected, and the Milnor number μ of §7 is the rank of $H_n(F)$. The Milnor number is zero if and only if the germ f has a regular point at $\mathbf{0}$ [Milnor 1, Corollary 7.3]. The *intersection pairing* $(\ ,\ )$ of F is the integral bilinear form $H_n(F) \times H_n(F) \to \mathbf{Z}$ defined by sending (x, y) to $(x' \cup y')[F]$, where x' and y' in $H^n(F, \partial F)$ are Lefschetz duals to x and y, and $[F]$ in $H_{2n}(F, \partial F)$ is the orientation class of F given by the underlying complex structure. The intersection pairing is symmetric if n is even, and skew symmetric if n is odd. For example, the germ $f(z_0, ..., z_n) = z_0^2 + ... + z_n^2$ has $H_n(F)$ a free cyclic group with generator e, and $(e, e) = 2(-1)^{n/2}$ or 0 according as n is even or odd. There are many methods of computing the intersection pairing in special cases.

By a tensor product theorem [Gabrielov 1; Sakamoto], the Milnor numbers of $f(z_0, ..., z_n)$ and $f(z_0, ..., z_n) + z_{n+1}^2 + ... + z_m^2$ are equal. The *quadratic form* of $f(z_0, ..., z_n)$ is defined to be the intersection pairing of the germ $f(z_0, ..., z_n) + z_{n+1}^2 + ... + z_m^2$ where $m \equiv 2 \pmod 4$. This is independent of the choice of m. For example, if $n \equiv 0 \pmod 4$ then the quadratic form of f is the negative of its intersection pairing; all this follows from the tensor product theorem. See also [Kauffmann and Neumann].

A germ f *topologically degenerates* to a germ g if there is an $\eta > 0$ and a family h_t of germs for $\{t \in \mathbf{C} : |t| < 2\eta\}$ with $h_\eta \sim f$, $h_0 \sim g$, and h_t of constant Milnor number for $t \neq 0$. Compare [Lê and Ramanujam]. Clearly right degeneracy implies topological degeneracy.

Lemma 12.1 [Tjurina 1, Theorem 1]. If f topologically degenerates to g, then there is an injection of $H_n(F_f)$ into $H_n(F_g)$ (where F_f is the Milnor fiber of f, and F_g is the Milnor fiber of g), and this injection preserves the intersection pairing. In particular, if g topologically degenerates to f as well, then the intersection pairings of f and g are isomorphic.

Characterization B5. The quadratic form of f is negative definite.

The equivalence of Characterizations B1 and B5 is proved in [Tjurina 1]. By explicit computation the quadratic forms of the germs in Table 2a are shown to be negative definite, and those of Table 2b are shown to be negative semi-definite. (In fact, the quadratic form of a germ in Table 2a is isomorphic to the intersection pairing of its minimal resolution, and the quadratic form of a germ of type $\tilde{E}_k$ in Table 2b is isomorphic to the quadratic form of E_k plus a two-dimensional zero form.) The result then follows from Proposition 10.1 and Lemma 12.1. When $n = 2$, the Milnor fiber F is in fact diffeomorphic to the minimal resolution M of $f^{-1}(0)$, since the singularity of $f^{-1}(0)$ is an absolutely isolated double point [Brieskorn 1, Theorem 4; Tjurina 1, Lemma 1].

When $n = 2$, the equivalence of Characterizations A2 and B5 follows from the following result [Durfee 2, Proposition 3.1].

THEOREM 12.2. *Twice the geometric genus* p *of* $f^{-1}(0)$ *equals the number of positive plus the number of zero diagonal elements in a diagonalization of the intersection pairing over the real numbers.*

The classification of germs according to signature of the quadratic form has been extended in [Arnold 3]; see also [Durfee 2, Proposition 3.3].

13. NEARBY MORSE FUNCTIONS

A *deformation* of a germ $f \in \mathscr{F}$ is a germ $g : \mathbf{C}^{n+1} \times \mathbf{C} \to \mathbf{C}$ with $g(z, 0) = f(z)$. Choose ε and δ for f as in §11. Then choose $\eta > 0$ such that for all $|t| < \eta$ and $|\delta'| \leqslant \delta$, the set $\{z \in \mathbf{C}^{n+1} : g(z, t) = \delta'\}$ intersects S_ε^{2n+1} transversally and the critical values of $g(z, t)$ for fixed t are less than δ in absolute value. A germ $\bar{f}$ is a *nearby Morse function* to f if $\bar{f}$ has only non-degenerate critical points in D_ε^{2n+2} and there is a deformation g and a t_0 with $|t_0| < \eta$ such that $\bar{f}(z) = g(z, t_0)$.

Characterization B6. There is a nearby Morse function to f with one or two critical values.

In fact, the nearby Morse function has one critical value if and only if f is right equivalent to A_2, since the quadratic form diagram is connected (§14). This surprising characterization is in [A'Campo 2II], where it is shown that Characterization B1 implies B6, and B6 implies B7 below.

14. Vanishing cycles

Let f be a germ in $\mathscr{F}$, and let $\tilde{f}$ be a nearby Morse function with μ distinct critical values $t_1, \ldots, t_u$ in the disk D_δ^2 of radius δ about 0 in **C**. A path α_i in $D_\delta^2 - \{t_1, \ldots, t_u\}$ from δ to t_i determines (up to sign) a *vanishing cycle* δ_i in $H_n(F)$. The self-intersection (δ_i, δ_i) is $2(-1)^{n/2}$ or 0 according as n is even or odd. Choose paths $\alpha_1, \ldots, \alpha_\mu$ in $D_\delta^2 - \{t_1, \ldots, t_\mu\}$ from δ to $t_1, \ldots, t_\mu$ respectively, such that the union of the images of the paths is a deformation retract of D_δ^2; then the corresponding vanishing cycles $\delta_1, \ldots, \delta_\mu$ are a basis of $H_n(F)$ [Brieskorn 4, Appendix]. The basis $\delta_1, \ldots, \delta_\mu$ is called an *ordered* (or *distinguished*) *basis of vanishing cycles* if $t_1, \ldots, t_\mu$ are ordered so that the loop going once counter-clockwise around the boundary of D_δ^2 is homotopic in $\pi_1(D_\delta^2 - \{t_1, \ldots, t_\mu\}, \delta)$ to the product $\beta_1 * \ldots * \beta_\mu$, where β_i is the loop going out α_i almost to t_i, around t_i counter-clockwise, and back along α_i. References for this are [Gabrielov 1, Lamotke, Durfee 1].

Choose an ordered basis of vanishing cycles $\delta_1, \ldots, \delta_\mu$ for the intersection pairing (,) of $f(z_0, \ldots, z_n) + z_{n+1}^2 + \ldots + z_m^2$, where $m \equiv 2 \pmod 4$ The *quadratic form diagram* of f with respect to the basis $\delta_1, \ldots, \delta_\mu$ has vertices $v_1, \ldots, v_\mu$ and edges from v_i to v_j if $(\delta_i, \delta_j) \neq 0$, weighted by (δ_i, δ_j) if $(\delta_i, \delta_j) \neq 1$. This diagram is connected [Lazzeri; Gabrielov 2]. It determines all the topological information in the singularity if $n \neq 2$ [Durfee 1]. There are a number of methods of computing these diagrams [A'Campo 2I; Gabrielov 3; Gusein-Zade]. The quadratic form diagrams of the germs of Table 2 are listed in column 5. Lemma 12.1 can be strengthened to show that if f topologically degenerates to g, then some quadratic form diagram for f is a subdiagram of some quadratic form diagram for g [Siersma, p. 82].

Characterization B7. There is an ordered basis of vanishing cycles for f such that the quadratic form diagram is a (weighted) tree.

It is shown in [A'Campo 2II] that Characterizations B1 and B7 are equivalent. In fact, the quadratic form diagrams for the germs in Table 2a are the same as the graph of their minimal resolutions (column 3 of Table 1).

15. The monodromy group

Let f be a germ in $\mathscr{F}$, and as above choose an ordered basis $\delta_1, \ldots, \delta_\mu$ of vanishing cycles for $H_m(F)$, where F is the Milnor fiber of

$$f(z_0, \ldots, z_n) + z_{n+1}^2 + \ldots + z_m^2$$

with $m \equiv 2 \pmod 4$. The *Picard-Lefshetz automorphisms* T_i of $H_m(F)$ for $i = 1, ..., \mu$ are defined by

$$T_i(x) = x + (\delta_i, x)\,\delta_i\,.$$

Another way of writing T_i is

$$T_i(x) = x - 2\,\frac{(\delta_i, x)}{(\delta_i, \delta_i)}\,\delta_i$$

which shows that T_i is a reflection in δ_i [Lamotke].

The *monodromy group* of f is the subgroup of the automorphism group of $H_m(F)$ generated by $T_1, ..., T_\mu$. This group depends only on f, since it may also be defined as a representation of the *braid group* of f, which is the fundamental group of the complement of the bifurcation diagram in the base space of the versal unfolding of f [Arnold 3, §2.8]. (Here is a direct proof that the monodromy group of f is independent of the choice of nearby Morse function $\bar{f}$ and paths $\alpha_1, ..., \alpha_\mu$: The set $D_\delta^2 - \{t_1, ..., t_\mu\}$ is the base space of a fiber bundle with fiber F, so $\pi_1(D_\delta^2 - \{t_1, ..., t_\mu\}, \delta)$ acts on $H_m(F)$. The image of β_i in Aut $H_m(F)$ is T_i; since $\beta_1, ..., \beta_\mu$ generate π_1, the monodromy group is the image of π_1 in Aut $H_m(F)$. Thus the monodromy group is independent of the choice of $\alpha_1, ..., \alpha_\mu$. It is independent of the choice of $\bar{f}$ since any two nearby Morse functions with μ distinct critical values can be joined by a family of such functions.)

Characterization B8. The monodromy group of f is finite.

Characterization B5 implies Characterization B8 since the automorphism group of any positive definite integral lattice is finite. In fact, the monodromy groups are precisely the Coxeter groups of the corresponding quadratic form diagram. Conversely, [Gabrielov 3] shows that if a germ f topologically degenerates to a germ g, then the monodromy group of f is a quotient of a subgroup of the monodromy group of g. Since the monodromy groups of the germs in Table 2b are infinite [Gabrielov 1], Proposition 10.1 shows that Characterization B8 implies Characterization B1.

16. Weighted homogeneous polynomials

A polynomial $g(z_0, ..., z_n)$ is *weighted homogeneous* if there are positive rational numbers $w_0, ..., w_n$ (the *weights*) such that $g(z_0, ..., z_n)$ may be written as a sum of monomials $z_0^{i_0} ... z_n^{i_n}$ with $i_0/w_0 + ... + i_n/w_n = 1$

[Milnor 1, p. 75; Orlik and Wagreich]. Another way of saying this is that if the variables z_i are weighted by $1/w_i$, then g is homogeneous of degree one, that is, $g(\lambda^{1/w_0}z_0, ..., \lambda^{1/w_n}z_n) = \lambda\, g(z_0, ..., z_n)$ for all complex numbers λ. All the germs in Table 1 are weighted homogeneous with weights as listed in Column 7. These germs remain weighted homogeneous upon adding squares of new variables, each weighted by 2. It is proved in [Saito 1, Lemma 4.3] that the weights of a germ g are uniquely determined (up to permutation) by the analytic isomorphism type of $g^{-1}(0)$.

Characterization B9. The germ $f^{-1}(0)$ is isomorphic to $g^{-1}(0)$, where g is a weighted homogeneous polynomial with weights w_i satisfying $w_0^{-1} + ... + w_n^{-1} > n/2$.

The equivalence of Characterizations B2 and B9 is proved in [Saito 2, Satz 2.11]. (The r there is $w_0^{-1} + ... + w_n^{-1}$.)

Appendix I

Nine Characterizations of Almost-Simple Critical Points (Simple Elliptic Singularities)

Almost-simple critical points can also be characterized in several ways. The nine characterizations presented in this appendix are analogues of some of those in the main text.

Theorem C. *Let $f(z_0, ..., z_n)$ with $n \geqslant 2$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that $\mathbf{0}$ is an isolated critical point. Then Characterizations C1 through C9 are equivalent.*

Characterization C1. The germ f is right equivalent to one of the germs in Table 2b.

Characterization C2. The germ f is contact equivalent to one of the germs in Table 2b.

The equivalence of these characterizations follows from Proposition 9.1.

A germ $f \in \mathfrak{m}$ is said to be *right* (or *contact*) *almost-simple* if f is not right (or contact) simple, but there is a neighborhood of f in $\mathfrak{m}$ intersecting only finitely many right (or contact) equivalence orbits of lower codimension in $\mathfrak{m}$.

Characterization C3. The germ f is right almost-simple.

Characterization C4. The germ f is contact almost-simple.

The equivalence of Characterizations C1 and C3 was conjectured by Milnor and is proved in [Arnold 3, §3.2.6] using Proposition 10.1. As for simple germs, Characterizations C3 and C4 are also equivalent.

Characterization C5. The quadratic form of f is not negative definite but negative semi-definite.

The equivalence of Characterization C1 and C5 proved in [Arnold 3] using §11 and the fact that the quadratic forms of the germs in Table 2c have one negative and one zero eigenvalue.

Characterization C6. The monodromy group of f is not finite but has polynomial growth.

For the notions of polynomial and exponential growth, see [Milnor 3]. This was conjectured by Milnor. It is shown in [Gabrielov 1] that the monodromy groups of the germs f in Table 2b have polynomial growth. A'Campo first proved that the monodromy groups of the germs in Table 2c have exponential growth. In Appendix II we present another proof of this fact due to Looijenga. Hence Characterizations C1 and C6 are equivalent.

Characterization C7. Assume $n = 2$. Conjecture: The local fundamental group of $f^{-1}(0)$ is not finite but has polynomial growth.

This was also conjectured by Milnor. It is shown in [Wagreich 2] that the local fundamental groups of the germs in Table 2b are nilpotent, and hence have polynomial growth. In fact, it is conjectured that if the germ $f(z_0, z_1, z_2)$ is not simple or almost-simple, then the local fundamental group of $f^{-1}(0)$ has exponential growth, and even contains a free non-abelian subgroup of finite index. See also [Orlik 1].

Characterization C8. Assume $n = 2$. The exceptional set in the minimal resolution of $f^{-1}(0)$ is a nonsingular elliptic curve E with $-3 \leqslant E^2 \leqslant -1$.

The equivalence of Characterizations C2 and C8 is proved in [Wagreich 2, p. 66; Saito 2, Theorem 1.9]. In fact, the zero loci of the germs $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ have minimal resolution as above with $E^2 = -3, -2$ and -1 respectively.

Characterization C9. The germ $f^{-1}(0)$ is isomorphic to $g^{-1}(0)$, where g is a weighted homogeneous polynomial with weights w_i satisfying $w_0^{-1} + \ldots + w_n^{-1} = n/2$.

The equivalence of Characterizations C2 and C9 is proved in [Saito 2, Satz 2.11]. In fact, the germs in Table 2b have the following weights:

GERM	WEIGHTS
P_8	(3, 3, 3)
X_9	(4, 4)
J_{10}	(3, 6)

Appendix II

The Monodromy Groups of the Minimal Hyperbolic Germs

Proposition. *The monodromy groups of the germs* P_9, X_{10}, *and* J_{11} *have exponential growth.*

In this appendix, we present an (unpublished) proof of this proposition due to E. Looijenga. In fact, we will show that these groups have $PSL(2, \mathbf{Z})$ as subquotient (quotient of a subgroup). We let $0(V)$ denote the orthogonal group of a **Z**- or **R**-module V equipped with a bilinear form.

Suppose G is a polyhedral graph whose edges are weighed by non-zero integers. By convention, the weight 1 is omitted. Let L_G denote the free **Z**-module generated by the vertices $v_1, \ldots, v_n$ of G. Define a symmetric bilinear form $(\,,\,)$ on L_G by setting $(v_i, v_i) = -2$, and $(v_i, v_j) = 0$ if there is no edge from v_i to v_j, otherwise equal to the weight on this edge. Conversely, given a symmetric integral bilinear form $(\,,\,)$ on a free module L with basis $\alpha_1, \ldots, \alpha_n$ with the property that $(\alpha_i, \alpha_i) = -2$ for all i, one associates a graph to it in the obvious way.

For $\alpha \in L_G$, let s_α (*reflection in* α) be the isometry of L_G defined by

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

for $\beta \in L_G$. The *reflection group* $\mathscr{R}(G)$ of the graph G is defined to be the subgroup of $0(L_G)$ generated by $s_{\alpha_1}, \ldots, s_{\alpha_n}$.

Example 1. Consider a reduced irreducible root system in a vector space V. Let $\alpha_1, \ldots, \alpha_n$ be a collection of simple roots, let L be the free $\mathbf{Z}$-module spanned by the α_i, and let $(,)$ be the negative of an invariant bilinear form [Serre, Chapter 5]. If $(\alpha_i, \alpha_i) = -2$ for all i, then the corresponding graph must be of type A_k, D_k, E_6, E_7 or E_8. The reflection group of these graphs equals the Weyl group, the group generated by reflections in all the roots [Serre, p. V-16]. Furthermore, the reflection group together with the generators $s_{\alpha_1}, \ldots, s_{\alpha_n}$ forms a Coxeter system [Bourbaki, p. 92]. (A *Coxeter system* is a group G, a collection of elements $g_1, \ldots, g_n$ and a symmetric integral $n \times n$ matrix $\{m_{ij}\}$ with $m_{ii} = 1$ and $2 \leqslant m_{ij} \leqslant \infty$ for $i \neq j$, with the property that G is isomorphic to the free group with generators $g_1, \ldots, g_n$ and relations $(g_i, g_j)^{m_{ij}} = 1$, for all i, j.)

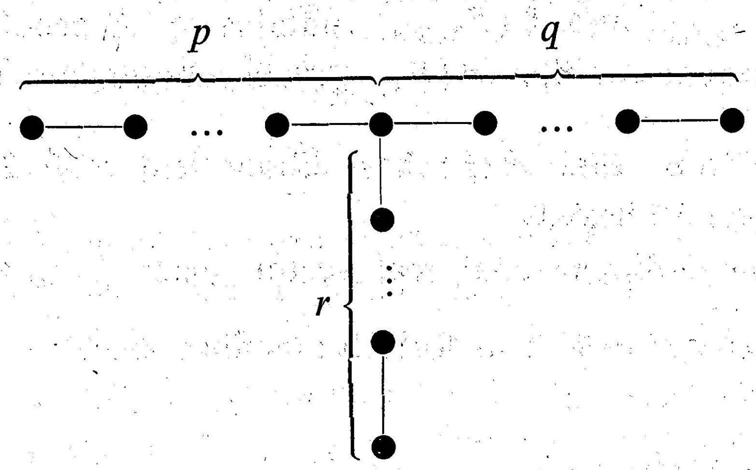
Example 2. The monodromy group of a germ f is the reflection group of a quadratic form diagram for f (Sections 13 and 14). When this diagram is a tree (which is only possible for the simple germs), its reflection group together with the generators $T_1, \ldots, T_\mu$ forms a Coxeter system. In general, this reflection group is not a Coxeter system [A'Campo 2, II, p. 403].

Lemma [Gabrielov 3]. If the graph G' is a subgraph of the graph G, then $\mathscr{R}(G')$ is a subquotient of $\mathscr{R}(G)$.

Proof. Let $\alpha'_1, \ldots, \alpha'_m$ be a basis of $L_{G'}$ corresponding to the vertices of G', let $\alpha_1, \ldots, \alpha_m$ be the corresponding elements in L_G, and extend this to a basis $\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n$ of L_G corresponding to the vertices of G. The map $\alpha'_i \to \alpha_i$ is an isometric embedding of $L_{G'}$ in L_G. Let $\mathscr{R}'$ be the subgroup of $\mathscr{R}(G)$ generated by $s_{\alpha_1}, \ldots, s_{\alpha_m}$; it has a presentation with these generators and certain relations. Any relation among these s_{α_i} holds also for $s_{\alpha_i} \mid L_{G'} = s_{\alpha'_i}$. Thus $\mathscr{R}'$ maps onto $\mathscr{R}(G')$.

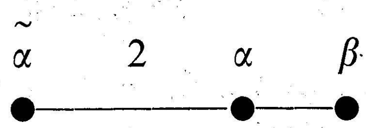
Fact. If a subquotient of a group G has exponential growth, then so does G.

Proof of Proposition. 1. A quadratic form diagram for the germs P_9, X_{10}, and J_{11} is given in column 5 of Table 2. These graphs contain a subgraph of the form $T_{3,3,4}$, $T_{2,4,5}$, and $T_{2,3,7}$ respectively, where $T_{p,q,r}$ is the graph

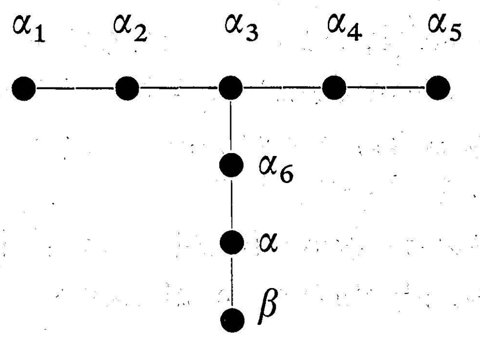


Hence it suffices to show that the reflection groups of these graphs have exponential growth.

Let Γ be the graph



with vertices corresponding to basis elements $\tilde{\alpha}$, α, β in L_Γ as indicated. We claim that $\mathscr{R}(T_{p,q,r})$ has $\mathscr{R}(\Gamma)$ as subquotient, for $(p, q, r) = (3, 3, 4)$, $(2, 4, 5)$, and $(2, 3, 7)$. Consider (for example) $T_{3,3,4}$, with vertices corresponding to basis elements $\alpha_i \in L_{T3,3,4}$ as indicated:



This contains the graph E_6. Let

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \in L_{T3,3,4}$$

be the largest root of E_6 [Bourbaki, p. 165]. Since all the roots of E_6 are the same length, $(\tilde{\alpha}, \tilde{\alpha}) = -2$. The lattice spanned by $\tilde{\alpha}$, α, and β has diagram Γ. The reflections s_α and s_β are in $\mathscr{R}(T_{3,3,4})$. We claim that $s_{\tilde{\alpha}}$ is in $\mathscr{R}(T_{3,3,4})$ as well: The restriction $s_{\tilde{\alpha}} \mid L_{E6}$ is in $\mathscr{R}(E_6)$, since E_6 is a root system. Hence $s_{\tilde{\alpha}} \mid L_{E6} = (s_{\alpha_{i(1)}} \circ \dots \circ s_{\alpha_{i(m)}}) \mid L_{E6}$ for some $1 \leqslant i(1)$, .., $i(m) \leqslant 6$.

Also, $s_{\tilde{\alpha}}$ and $s_{\alpha_{i(1)}} \circ \ldots \circ s_{\alpha_{i(m)}}$ are both the identity when restricted to the orthogonal complement of $L_{E_6} \otimes \mathbf{R}$ in $L_{T_{3,3,4}} \otimes \mathbf{R}$. Thus $s_{\tilde{\alpha}} = s_{\alpha_{i(1)}} \circ \ldots \circ s_{\alpha_{i(m)}}$, and $\mathscr{R}(T_{3,3,4})$ contains $s_{\tilde{\alpha}}$. A proof similar to that of the lemma then shows that $\mathscr{R}(T_{3,3,4})$ has subquotient $\mathscr{R}(\Gamma)$.

3. Next we show that $\mathscr{R}(\Gamma)$ has subquotient $PSL(2, \mathbf{Z})$. This uses [E. Artin, Chapter V] heavily.

Let V be the 3-dimensional real vector space $L_\Gamma \otimes \mathbf{R}$. The bilinear form $(\,,)$ of Γ extends to V. This form is indefinite since $\tilde{\alpha} + \alpha$ has length 0. Let

$$0'(V) = \{f \in 0(V)\colon \det f = 1 \text{ and spinor norm of } f \text{ equal to } 1\,\mathbf{R}^{*2}\}.$$

Since V is indefinite, it is known [E. Artin, p. 200] that

$$0'(V) \xrightarrow{\cong} PSL(2, \mathbf{R})\,. \tag{1}$$

Since $PSL(2, \mathbf{R})$ contains $PSL(2, \mathbf{Z})$ as a subgroup, the idea is to find elements of $\mathscr{R}(\Gamma) \subset 0(\Gamma)$ which are in $0'(V)$ and map to generators of $PSL(2, \mathbf{Z})$. The standard generators of $PSL(2, \mathbf{Z})$ are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with relations $S^2 = (ST)^3 = 1$. By inspection, it is found that the elements $s_{\tilde{\alpha}}\, s_\beta$ and $s_\beta\, s_\alpha$ of $0(L)$ satisfy $(s_{\tilde{\alpha}} s_\beta)^2 = (s_\beta s_\alpha)^3 = 1$, and have determinant equal to 1 and spinor norm equal to $1\mathbf{R}^{*2}$. Therefore we would like to choose the isomorphism (1) such that $s_{\tilde{\alpha}}\, s_\beta$ maps to S, and $s_{\tilde{\alpha}}\, s_\alpha = (s_{\tilde{\alpha}} s_\beta)^{-1} (s_\beta s_\alpha)$ maps to $S^{-1}(ST) = T$.

The isomorphism (1) is done in two steps. First, let $D_0(V)$ be the elements of the Clifford algebra of V of norm 1; then [E. Artin, p. 199]

$$D_0(V)/\{\pm 1\} \xrightarrow{\cong} 0'(V)\,. \tag{2}$$

We do not need to know exactly what this isomorphism is, but only that

$$v \circ w \to s_v s_w$$

for elements v, w in V regarded as a subspace of the Clifford algebra, and $v \circ w$ their product. Hence under the above isomorphisms

$$\frac{1}{2}\tilde{\alpha} \circ \beta \mapsto s_{\tilde{\alpha}} s_\beta\,, \qquad \frac{1}{2}\tilde{\alpha} \circ \alpha \mapsto s_{\tilde{\alpha}} s_\alpha\,. \tag{3}$$

Secondly [E. Artin, p. 199],

$$D_0(V)/\{\pm 1\} \cong PSL(2,\mathbf{R}). \tag{4}$$

We examine this more closely. Let

$$A_1 = \sqrt{2}(\alpha+\tilde{\alpha}+\beta/2), \quad A_2 = \tilde{\alpha}/\sqrt{2}, \quad A_3 = \beta/\sqrt{2}.$$

Then A_1, A_2, A_3 is an orthogonal basis of V, and the matrix of $(\ ,\ )$ with respect to this basis is the diagonal matrix $\langle +1, -1, -1\rangle$. Let $C^+(V)$ be the subspace of the Clifford algebra of V spanned by the elements of even grading; $C^+(V)$ is generated by 1, i_1, i_2, i_3, where $i_1 = A_2 \circ A_3$, $i_2 = A_3 \circ A_1$, and $i_3 = A_1 \circ A_2$, and has multiplication table as in [E. Artin, top of p. 200] with $a = -1$. The map

$$C^+(V) \to M(2,\mathbf{R})$$

(where $M(2,\mathbf{R})$ is the algebra of all 2×2 matrices over $\mathbf{R}$) defined by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$i_2 \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i_3 \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

is an isomorphism. (This is slightly different from the isomorphism of [E. Artin, p. 200].), and the restriction of this map to $D_0(V)$ gives the isomorphism (4). Furthermore,

$$\frac{1}{2}\tilde{\alpha}\circ\beta = i_1 \mapsto S\,, \quad \frac{1}{2}\tilde{\alpha}\circ\alpha = 1 - \frac{1}{2}(i_1+i_3) \mapsto T \tag{5}$$

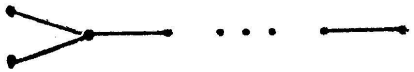
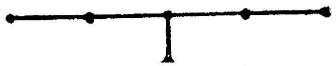
under this isomorphism. Combining isomorphisms (2) and (4) gives isomorphism (1), and (3) and (5) show that

$$s_{\tilde{\alpha}}\, s_\beta \mapsto S\,, \qquad s_{\tilde{\alpha}}\, s_\alpha \mapsto T$$

under isomorphism (1). Thus $\mathscr{R}(\Gamma)$ maps onto $PSL(2, \mathbf{Z})$, and hence has $PSL(2, \mathbf{Z})$ as subquotient.

4. Finally, $PSL(2, \mathbf{Z})$ is isomorphic to the free product $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/3\mathbf{Z})$ [Serre, ch. 7; Lehner, p. 59], which has exponential growth.

TABLE 1 RATIONAL DOUBLE POINTS

(1) $f(x, y, z)$	(2) (p, q, r)	(3) Dual Graph of Resolution	(4) Name	(5) Subgroup G of S^3	(6) Order of G	(7) Weights
$x^{k+1}+y^2+z^2$	$(1, 1, k)$	 k vertices	A_k $k \geqslant 1$	cyclic	$k+1$	$(k+1, 2, 2)$
$x^{k-1}+xy^2+z^2$	$(2, 2, k-2)$	 k vertices	D_k $k \geqslant 4$	binary dihedral	$4(k-2)$	$(k-1, 2(k-1)/(k-2), 2)$
$x^3+y^4+z^2$	$(2, 3, 3)$		E_6	binary tetrahedral	24	$(3, 4, 2)$
$x^3+xy^3+z^2$	$(2, 3, 4)$		E_7	binary octahedral	48	$(3, 9/2, 2)$
$x^3+y^5+z^2$	$(2, 3, 5)$		E_8	binary icosahedral	120	$(3, 5, 2)$

Column 1 lists the germ $f(x, y, z)$. Column 3 lists the dual graph of the minimal resolution of $f^{-1}(0)$. The name of the graph is given in column 4. Each graph is of type $T_{p,q,r}$, where (p, q, r) are listed in column 2. Every vertex of the graph represents a nonsingular rational curve of self-intersection -2. The analytic set $f^{-1}(0)$ is isomorphic to $\mathbf{C}^2/G$, where G is the finite subgroup of S^3 listed in column 5. Each germ f is weighted homogeneous, with weights as listed in column 7.

TABLE 2 GERMS OF LOW MILNOR NUMBER

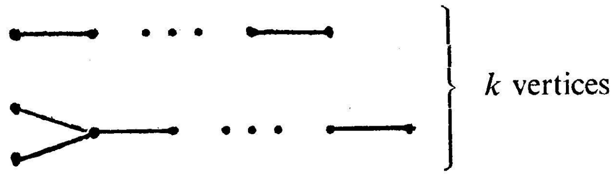
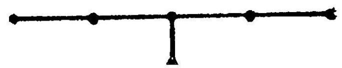
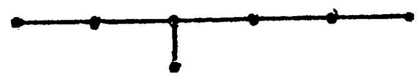
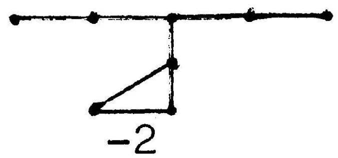
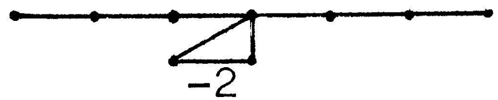
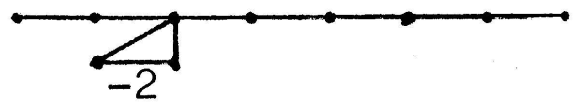
(1)	(2) Name	(3) μ	(4) $f(z_0, \ldots, z_n)$	(5) Quadratic form diagram
a	$A_k,\ k \geqslant 1$	k	z_0^{k+1}	
	$D_k,\ k \geqslant 4$	k	$z_0^{k+1} + z_0 z_1^2$	
(simple germs)	E_6	6	$z_0^3 + z_1^4$	
	E_7	7	$z_0^3 + z_0 z_1^3$	
	E_8	8	$z_0^3 + z_1^5$	
b	P_8 or $\tilde{E}_6$	8	$z_0^3 + z_1^2 z_2 + a z_0 z_2^2 + b z_2^3,\ 4a^3 + 27b^2 \neq 0$	
(almost-simple germs)	X_9 or $\tilde{E}_7$	9	$z_0 z_1 (z_0 - z_1)(z_0 - a z_1),\ a \neq 0, 1$	
	J_{10} or $\tilde{E}_8$	10	$z_0 (z_0 - z_1^2)(z_0 - a z_1^2),\ a \neq 0, 1$	

TABLE 2 *(continuation)*

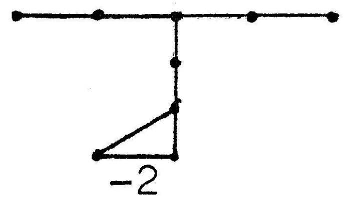
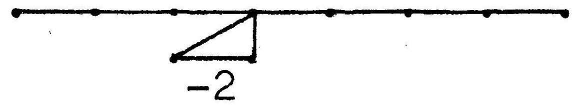
c	P_9 or $T_{3,3,4}$	9	$z_0^2 z_2 + z_1^3 + z_1^2 z_2 + a z_2^4 , \ a \neq 0$	
(minimal hyperbolic germs)	X_{10} or $T_{2,4,5}$	10	$z_0^4 + z_0^2 z_1^2 + a z_1^5 , \ a \neq 0$	
	J_{11} or $T_{2,3,7}$	11	$z_0^3 + {}_\varepsilon z_0^2 z_1^2 + a z_1^7 , \ a \neq 0 , \ {}_\varepsilon = \pm 1$	

Table 2 is divided into parts a, b, and c as indicated in column 1. These parts are referred to in the text as Table 2a, etc. Column 4 contains the equation of the germ; to each equation must be added the quadratic form $z_r^2 + \ldots + z_n^2$ in the variables $z_r, \ldots, z_n$ not occurring in the equation. Column 2 gives the type of the germ in Arnold's notation, column 3 gives its Milnor number μ, and column 5 gives a quadratic form diagram.

TABLE 3

SIMPLIFICATION TABLE

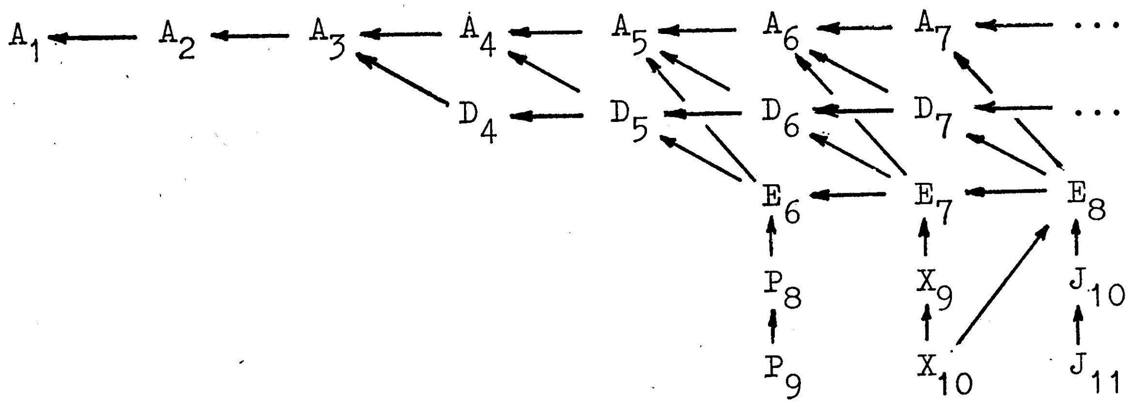


Table 3 lists some (but not all) of the simplifications that occur among the germs of Table 2.

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