# 1. Complex analytic spaces

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fifteen characterizations, since Characterization A1 coincides with Characterization B2.

Most of the characterizations of Part B are shown to be equivalent to Characterization B1. Other links between the two sets of characterizations are provided by Theorem 12.2, which shows that Characterizations A2 and B5 are equivalent, and a recent result (Theorem 11.1) partially connecting Characterizations A2 and B3. Part B also contains a summary of pertinent work of Mather and Arnold.

There are two appendices. The first gives nine characterizations of simple elliptic singularities and almost-simple critical points. They are the next most reasonable class of singularities after rational double points, and can be characterized as being "infinite but not too infinite". All remaining singularities are "very infinite" in various senses. The second appendix contains Looijenga's proof that the monodromy group of the minimal hyperbolic germs has exponential growth.

This paper is an expanded version of a series of lectures given at the University of Maryland in the spring of 1976, and I thank the department of mathematics for its hospitality. The lectures were inspired by an unpublished talk given by E. Brieskorn at the American Mathematical Society Summer Institute in Algebraic Geometry in Arcata (1974). I also thank E. Looijenga and J. Wahl for helpful comments.

## A. Seven characterizations of rational double points

Theorem A. *Let* $f(x, y, z)$ *be the germ at the origin* $\mathbf{0}$ *of a complex analytic function, and suppose that* $f(\mathbf{0}) = 0$ *and that the origin is an isolated critical point of* $f$. *Then characterizations A1 through A7 (which are listed below) are equivalent.*

### 1. Complex analytic spaces

Let $V$ be the germ at $\mathbf{v}$ of a normal two-dimensional complex analytic space with a singularity at $\mathbf{v}$. (The definitions of these terms can be found in [Laufer 1].) For example, $V$ could be $f^{-1}(0)$, where $f$ is as in the hypotheses of Theorem A. Conversely, if $V$ is embedded in $\mathbf{C}^3$ with $\mathbf{v}$ the origin, there is a germ $f$ as above such that $V$ is isomorphic to $f^{-1}(0)$ [Gunning and Rossi, p. 113]. The singularity is isolated since $V$ is normal. Two such germs $V$ and $W$ embedded in $\mathbf{C}^n$ at the origin are *isomorphic* if there is a germ of an analytic automorphism of $\mathbf{C}^n$ fixing the origin and taking $V$ to $W$.

*Characterization A1.* The analytic set $f^{-1}(0)$ is isomorphic to the zero locus of one of the functions listed in column 1 of Table 1.

## 2. Rational singularities

A *resolution* of a germ of a normal surface singularity $V$ as above is a complex analytic manifold $M$ and an analytic map $\pi: M \to V$ that is surjective and proper (compact fibers) such that its restriction to $M - \pi^{-1}(\mathbf{v})$ is an analytic isomorphism, and $M - \pi^{-1}(\mathbf{v})$ is dense in $M$. Resolutions exist, and can be computed with a certain amount of effort. The article [Lipman 2] contains a general discussion of resolutions, and [Laufer 1] and [Hirzebruch, Neumann, and Koh, §9] give a detailed method with examples.

Among all resolutions there is a *minimal resolution* $\pi: M \to V$ that has the following universal mapping property: Given any other resolution $\pi': M' \to V$, there is a unique map $\rho: M' \to M$ with $\pi' = \pi \circ \rho$.

The *geometric genus p* of $V$ is the dimension of the complex vector space $H^1(M, \mathcal{O}_M)$, where $M$ is any resolution of $V$, and $\mathcal{O}_M$ is the sheaf of holomorphic functions on $M$ [Artin; Wagreich 1, §1.4; Brieskorn 2; Laufer 2]. ($V$ is assumed Stein.) This number is finite, and independent of the choice of resolution. It may alternately be defined as the dimension of the stalk at the origin of the sheaf $R^1 \pi_* \mathcal{O}_M$ on $V$. The idea behind this definition is that $M$ is a collection of "thickened" curves, and that the genus of a curve $X$ is the dimension of $H^1(X, \mathcal{O}_X)$. For example, $H^1(M, \mathcal{O}_M) = 0$ if $M$ is the total space of a line bundle over a curve of genus zero. On the other hand, dim $H^1(M, \mathcal{O}_M) = k(k-1)(k-2)/6$ if $M$ is a line bundle of Chern class $-k$ over a curve of genus $(k-1)(k-2)/2$ (the minimal resolution of $f(x, y, z) = x^k + y^k + z^k$). In terms of $V$ alone, $p$ is the dimension of the space of holomorphic two-forms on $V - \mathbf{v}$ divided by square-integrable forms [Laufer 2, Theorem 3.4]. Another formula for $p$ in terms of topological invariants of the resolution $M$ and the nearby fiber $F$ (see §11) is given in [Laufer 6].

The analytic set $V$ has a *rational* singularity if $p = 0$. A rational singularity embeds in codimension 1 if and only if it is a double point (its local ring is of multiplicity two) [Artin, Corollary 6].