

## 6. The local fundamental group

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Proposition 5.2 shows that characterizations A5' and A1 are equivalent. Clearly Characterization A5' implies A5; since A5 implies A2, they are all equivalent.

**COROLLARY 5.3.** *Let  $G$  be a small finite subgroup of  $GL(2, \mathbb{C})$ . Then  $G \subset SL(2, \mathbb{C})$  if and only if  $\mathbb{C}^2/G$  embeds in codimension one.*

This corollary follows from the above case-by-case analysis. J. Wahl points out that it is also possible to prove it directly, using the following two facts:

*Fact 1.* Let  $G$  be a small finite subgroup of  $GL(2, \mathbb{C})$ . Then  $G \subset SL(2, \mathbb{C})$  if and only if the singularity of  $\mathbb{C}^2/G$  is Gorenstein.

This is a special case of [Watanabe]. A germ of a normal two-dimensional complex space is *Gorenstein* if there is a nowhere-vanishing holomorphic two-form on its regular points.

*Fact 2.* Let  $V$  be the germ at  $\mathfrak{v}$  of a two-dimensional rational singularity. Then  $V$  is Gorenstein if and only if  $V$  embeds in codimension 1.

*Proof.* Any singularity embedded in codimension one is Gorenstein. Conversely, suppose  $V$  is Gorenstein. Let  $\pi: M \rightarrow V$  be the minimal resolution of  $V$ , and let  $E_1 \cup \dots \cup E_s = \pi^{-1}(\mathfrak{v})$  be its exceptional set as in Section 3. Since  $V$  is Gorenstein, there is a divisor  $K$  on  $M$  (the *canonical class*) satisfying the adjunction formula. Furthermore  $K \cdot E_i \geq 0$  for all  $i$  since the resolution is minimal, so  $K \leq 0$  [Artin, bottom of p. 130]. If  $K < 0$ , then  $-K > 0$ , so arithmetic genus  $p$  of  $-K$  satisfies  $p(-K) \leq 0$  [Artin, Proposition 1]. On the other hand,  $p(-K) = 1 - \chi(-K) = 1$  by the Riemann-Roch Theorem, a contradiction. Hence  $K = 0$ . Thus  $K \cdot E_i = 0$  for all  $i$ , so  $V$  is a double point and embeds in codimension one, as in the proof that Characterization A3 implies Characterization A2.

## 6. THE LOCAL FUNDAMENTAL GROUP

Let  $V$  be the germ of a normal two-dimensional complex analytic space with an isolated singularity at  $\mathfrak{v}$ . Without loss of generality, we may assume that  $V$  is a *good neighborhood* of  $\mathfrak{v}$ , that is, that there is a neighborhood basis  $V_i$  of  $\mathfrak{v}$  in  $V$  such that each  $V_i - \mathfrak{v}$  is a deformation retract of  $V - \mathfrak{v}$  [Prill]. The *local fundamental group* of  $V$  at  $\mathfrak{v}$  is then defined as  $\pi_1(V - \mathfrak{v})$ . This group is trivial if and only if  $V$  is nonsingular at  $\mathfrak{v}$  [Mumford].

PROPOSITION 5.1 (bis). *The following statement is equivalent to those listed above.*

(d) *The local fundamental group of  $V$  is finite.*

It is shown in [Prill, p. 381; Brieskorn 2, p. 344] that conditions (a) and (d) are equivalent.

*Characterization A6.* The local fundamental group of  $f^{-1}(0)$  is finite. Thus Characterizations A5 and A6 are equivalent.

There is an algorithm for computing the local fundamental group of  $V$  from a resolution [Mumford], and singularities  $V$  with finite, nilpotent and solvable local fundamental group have been classified [Brieskorn 2; Wagreich 2]. When  $V$  is a complete intersection, this classification is particularly simple [Durfee 2, Proposition 3.3].

## 7. VOLUME

Let  $f(x, y, z)$  be the germ at the origin  $\mathbf{0}$  of a complex analytic function, and suppose that  $f(\mathbf{0}) = 0$  and that the origin is an isolated critical point of  $f$ . There is an  $\varepsilon > 0$  such that  $f^{-1}(0)$  intersects all spheres of radius  $\varepsilon'$  about  $\mathbf{0}$  transversally for  $0 < \varepsilon' \leq \varepsilon$ . (See Section 12.) For  $t \in \mathbf{C}$ , let

$$V_t = f^{-1}(t) \cap D_\varepsilon^6$$

where  $D_\varepsilon^6$  is the closed disk of radius  $\varepsilon$  about  $\mathbf{0}$ . The function  $f(x, y, z)$  takes the constant value  $t$  on  $V_t$ , so  $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \equiv 0$  there. Hence a nowhere-vanishing holomorphic two-form  $\omega_t$  on  $V_t$  may be defined by the equivalent expressions

$$\omega_t = \frac{dy \wedge dz}{\partial f / \partial x} = \frac{dz \wedge dx}{\partial f / \partial y} = \frac{dx \wedge dy}{\partial f / \partial z},$$

*Characterization A7.* The integral  $\int_{V_0} \omega_0 \wedge \bar{\omega}_0$  is finite.

Note that the form  $\omega_0 \wedge \bar{\omega}_0$  takes positive real values. The equivalence of Characterizations A2 and A7 is due to Laufer, and follows easily from his expression for the geometric genus in terms of forms [Laufer 2, Corollary 3.6].

A different formulation of this characterization is due to E. Looijenga (unpublished): Let  $\Delta(r) = \{t \in \mathbf{C} : t < r\}$ , let

$$X(r) = f^{-1}(\Delta(r)) \cap D_\varepsilon^6$$

and let  $\text{vol}(X(r))$  be its volume in  $\mathbf{C}^3$ .