15. The monodromy group

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14. Vanishing cycles

Let f be a germ in $\mathscr{F}$, and let $\bar{f}$ be a nearby Morse function with μ distinct critical values $t_1, ..., t_u$ in the disk D_δ^2 of radius δ about 0 in **C**. A path α_i in $D_\delta^2 - \{t_1, ..., t_u\}$ from δ to t_i determines (up to sign) a *vanishing cycle* δ_i in $H_n(F)$. The self-intersection (δ_i, δ_i) is $2(-1)^{n/2}$ or 0 according as n is even or odd. Choose paths $\alpha_1, ..., \alpha_\mu$ in $D_\delta^2 - \{t_1, ..., t_\mu\}$ from δ to $t_1, ..., t_\mu$ respectively, such that the union of the images of the paths is a deformation retract of D_δ^2; then the corresponding vanishing cycles $\delta_1, ..., \delta_\mu$ are a basis of $H_n(F)$ [Brieskorn 4, Appendix]. The basis $\delta_1, ..., \delta_\mu$ is called an *ordered* (or *distinguished*) *basis of vanishing cycles* if $t_1, ..., t_\mu$ are ordered so that the loop going once counter-clockwise around the boundary of D_δ^2 is homotopic in $\pi_1(D_\delta^2 - \{t_1, ..., t_\mu\}, \delta)$ to the product $\beta_1 * ... * \beta_\mu$, where β_i is the loop going out α_i almost to t_i, around t_i counter-clockwise, and back along α_i. References for this are [Gabrielov 1, Lamotke, Durfee 1].

Choose an ordered basis of vanishing cycles $\delta_1, ..., \delta_\mu$ for the intersection pairing (,) of $f(z_0, ..., z_n) + z_{n+1}^2 + ... + z_m^2$, where $m \equiv 2 \pmod 4$ The *quadratic form diagram* of f with respect to the basis $\delta_1, ..., \delta_\mu$ has vertices $v_1, ..., v_\mu$ and edges from v_i to v_j if $(\delta_i, \delta_j) \neq 0$, weighted by (δ_i, δ_j) if $(\delta_i, \delta_j) \neq 1$. This diagram is connected [Lazzeri; Gabrielov 2]. It determines all the topological information in the singularity if $n \neq 2$ [Durfee 1]. There are a number of methods of computing these diagrams [A'Campo 2I; Gabrielov 3; Gusein-Zade]. The quadratic form diagrams of the germs of Table 2 are listed in column 5. Lemma 12.1 can be strengthened to show that if f topologically degenerates to g, then some quadratic form diagram for f is a subdiagram of some quadratic form diagram for g [Siersma, p. 82].

Characterization B7. There is an ordered basis of vanishing cycles for f such that the quadratic form diagram is a (weighted) tree.

It is shown in [A'Campo 2II] that Characterizations B1 and B7 are equivalent. In fact, the quadratic form diagrams for the germs in Table 2a are the same as the graph of their minimal resolutions (column 3 of Table 1).

15. The monodromy group

Let f be a germ in $\mathscr{F}$, and as above choose an ordered basis $\delta_1, ..., \delta_\mu$ of vanishing cycles for $H_m(F)$, where F is the Milnor fiber of

$$f(z_0, ..., z_n) + z_{n+1}^2 + ... + z_m^2$$

with $m \equiv 2 \pmod 4$. The *Picard-Lefshetz automorphisms* T_i of $H_m(F)$ for $i = 1, ..., \mu$ are defined by

$$T_i(x) = x + (\delta_i, x)\,\delta_i\,.$$

Another way of writing T_i is

$$T_i(x) = x - 2\,\frac{(\delta_i, x)}{(\delta_i, \delta_i)}\,\delta_i$$

which shows that T_i is a reflection in δ_i [Lamotke].

The *monodromy group* of f is the subgroup of the automorphism group of $H_m(F)$ generated by $T_1, ..., T_\mu$. This group depends only on f, since it may also be defined as a representation of the *braid group* of f, which is the fundamental group of the complement of the bifurcation diagram in the base space of the versal unfolding of f [Arnold 3, §2.8]. (Here is a direct proof that the monodromy group of f is independent of the choice of nearby Morse function $\bar{f}$ and paths $\alpha_1, ..., \alpha_\mu$: The set $D^2_\delta - \{t_1, ..., t_\mu\}$ is the base space of a fiber bundle with fiber F, so $\pi_1(D^2_\delta - \{t_1, ..., t_\mu\}, \delta)$ acts on $H_m(F)$. The image of β_i in Aut $H_m(F)$ is T_i; since $\beta_1, ..., \beta_\mu$ generate π_1, the monodromy group is the image of π_1 in Aut $H_m(F)$. Thus the monodromy group is independent of the choice of $\alpha_1, ..., \alpha_\mu$. It is independent of the choice of $\bar{f}$ since any two nearby Morse functions with μ distinct critical values can be joined by a family of such functions.)

Characterization B8. The monodromy group of f is finite.

Characterization B5 implies Characterization B8 since the automorphism group of any positive definite integral lattice is finite. In fact, the monodromy groups are precisely the Coxeter groups of the corresponding quadratic form diagram. Conversely, [Gabrielov 3] shows that if a germ f topologically degenerates to a germ g, then the monodromy group of f is a quotient of a subgroup of the monodromy group of g. Since the monodromy groups of the germs in Table 2b are infinite [Gabrielov 1], Proposition 10.1 shows that Characterization B8 implies Characterization B1.

16. Weighted homogeneous polynomials

A polynomial $g(z_0, ..., z_n)$ is *weighted homogeneous* if there are positive rational numbers $w_0, ..., w_n$ (the *weights*) such that $g(z_0, ..., z_n)$ may be written as a sum of monomials $z_0^{i_0} \dots z_n^{i_n}$ with $i_0/w_0 + \dots + i_n/w_n = 1$