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[Milnor 1, p. 75; Orlik and Wagreich]. Another way of saying this is that if the variables z_i are weighted by $1/w_i$, then g is homogeneous of degree one, that is, $g(\lambda^{1/w_0}z_0, \ldots, \lambda^{1/w_n}z_n) = \lambda\, g(z_0, \ldots, z_n)$ for all complex numbers λ. All the germs in Table 1 are weighted homogeneous with weights as listed in Column 7. These germs remain weighted homogeneous upon adding squares of new variables, each weighted by 2. It is proved in [Saito 1, Lemma 4.3] that the weights of a germ g are uniquely determined (up to permutation) by the analytic isomorphism type of $g^{-1}(0)$.

Characterization B9. The germ $f^{-1}(0)$ is isomorphic to $g^{-1}(0)$, where g is a weighted homogeneous polynomial with weights w_i satisfying $w_0^{-1} + \ldots + w_n^{-1} > n/2$.

The equivalence of Characterizations B2 and B9 is proved in [Saito 2, Satz 2.11]. (The r there is $w_0^{-1} + \ldots + w_n^{-1}$.)

APPENDIX I

NINE CHARACTERIZATIONS OF ALMOST-SIMPLE CRITICAL POINTS (SIMPLE ELLIPTIC SINGULARITIES)

Almost-simple critical points can also be characterized in several ways. The nine characterizations presented in this appendix are analogues of some of those in the main text.

THEOREM C. *Let $f(z_0, \ldots, z_n)$ with $n \geqslant 2$ be the germ at the origin* $\mathbf{0}$ *of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that* $\mathbf{0}$ *is an isolated critical point. Then Characterizations C1 through C9 are equivalent.*

Characterization C1. The germ f is right equivalent to one of the germs in Table 2b.

Characterization C2. The germ f is contact equivalent to one of the germs in Table 2b.

The equivalence of these characterizations follows from Proposition 9.1.

A germ $f \in \mathfrak{m}$ is said to be *right* (or *contact*) *almost-simple* if f is not right (or contact) simple, but there is a neighborhood of f in $\mathfrak{m}$ intersecting only finitely many right (or contact) equivalence orbits of lower codimension in $\mathfrak{m}$.

Characterization C3. The germ f is right almost-simple.

Characterization C4. The germ f is contact almost-simple.

The equivalence of Characterizations C1 and C3 was conjectured by Milnor and is proved in [Arnold 3, §3.2.6] using Proposition 10.1. As for simple germs, Characterizations C3 and C4 are also equivalent.

Characterization C5. The quadratic form of f is not negative definite but negative semi-definite.

The equivalence of Characterization C1 and C5 proved in [Arnold 3] using §11 and the fact that the quadratic forms of the germs in Table 2c have one negative and one zero eigenvalue.

Characterization C6. The monodromy group of f is not finite but has polynomial growth.

For the notions of polynomial and exponential growth, see [Milnor 3]. This was conjectured by Milnor. It is shown in [Gabrielov 1] that the monodromy groups of the germs f in Table 2b have polynomial growth. A'Campo first proved that the monodromy groups of the germs in Table 2c have exponential growth. In Appendix II we present another proof of this fact due to Looijenga. Hence Characterizations C1 and C6 are equivalent.

Characterization C7. Assume $n = 2$. Conjecture: The local fundamental group of $f^{-1}(0)$ is not finite but has polynomial growth.

This was also conjectured by Milnor. It is shown in [Wagreich 2] that the local fundamental groups of the germs in Table 2b are nilpotent, and hence have polynomial growth. In fact, it is conjectured that if the germ $f(z_0, z_1, z_2)$ is not simple or almost-simple, then the local fundamental group of $f^{-1}(0)$ has exponential growth, and even contains a free non-abelian subgroup of finite index. See also [Orlik 1].

Characterization C8. Assume $n = 2$. The exceptional set in the minimal resolution of $f^{-1}(0)$ is a nonsingular elliptic curve E with $-3 \leqslant E^2 \leqslant -1$.

The equivalence of Characterizations C2 and C8 is proved in [Wagreich 2, p. 66; Saito 2, Theorem 1.9]. In fact, the zero loci of the germs $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ have minimal resolution as above with $E^2 = -3, -2$ and -1 respectively.

Characterization C9. The germ $f^{-1}(0)$ is isomorphic to $g^{-1}(0)$, where g is a weighted homogeneous polynomial with weights w_i satisfying $w_0^{-1} + ... + w_n^{-1} = n/2$.

The equivalence of Characterizations C2 and C9 is proved in [Saito 2, Satz 2.11]. In fact, the germs in Table 2b have the following weights:

GERM	WEIGHTS
P_8	(3, 3, 3)
X_9	(4, 4)
J_{10}	(3, 6)

Appendix II

The Monodromy Groups of the Minimal Hyperbolic Germs

PROPOSITION. *The monodromy groups of the germs P_9, X_{10}, and J_{11} have exponential growth.*

In this appendix, we present an (unpublished) proof of this proposition due to E. Looijenga. In fact, we will show that these groups have $PSL(2, \mathbf{Z})$ as subquotient (quotient of a subgroup). We let $0(V)$ denote the orthogonal group of a **Z**- or **R**-module V equipped with a bilinear form.

Suppose G is a polyhedral graph whose edges are weighed by non-zero integers. By convention, the weight 1 is omitted. Let L_G denote the free **Z**-module generated by the vertices $v_1, ..., v_n$ of G. Define a symmetric bilinear form $(\ ,\ )$ on L_G by setting $(v_i, v_i) = -2$, and $(v_i, v_j) = 0$ if there is no edge from v_i to v_j, otherwise equal to the weight on this edge. Conversely, given a symmetric integral bilinear form $(\ ,\ )$ on a free module L with basis $\alpha_1, ..., \alpha_n$ with the property that $(\alpha_i, \alpha_i) = -2$ for all i, one associates a graph to it in the obvious way.