AN EQUIVARIANT SETTING OF THE MORSE THEORY

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by Raoul BOTT [2]

This being a "Feiertag" in the true sense of the word, and mine being the last lecture—as well as being so far out of line of the topic of this symposium—you will permit me a few words of a general nature; an explanation so to speak, why I have been invited to these festivities.

I expect we all find periods in our life when it seems that Providence is at the helm and all that is asked of us is to give it free reign. To me the year 1949-50 is such a period and it is as clearly etched in my mind now as it was ten, -twenty, -thirty! years ago. For then, quite providentially at the last possible moment I was asked to the Institute for Advanced Study at Princeton a brash brand new Ph.D from Carnegie Tech, and I had the whole miraculous world of pure mathematics burst upon me. At that time Princeton was inhabited by giants. One saw Einstein and Gödell stroll arm in arm—I recall a lecture by Dirac with Einstein, Pauli, von Neumann, Hermann Weyl and Oppenheimer in the audience—and it was my great good fortune that on this exciting trip providence had also provided me with a wonderful guide, a very dear friend, and at the same time a quite exacting tutor. That young man is our birthday-child of today. I of course have a store of anecdotes from that time, and maybe I can divulge some of them later tonight. Here let me mention only one. I bombarded Ernst with so many stupid questions that in desperation he finally imposed a fine of 25 cents (!) on any conjecture he could disprove in less than five minutes. This should give you some idea of the inflation of the past thirty years and also help to explain Ernst's vast fortune at this time.

Of course I had many other teachers there: Reidemeister, Steenrod—Ernst and I attended his course on Fiber-bundles—and also friends, many of whom I am delighted to see here today: de Rham, Erdös, Beno Eckmann,

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Chandrasekharan. How wonderful to see you all—and how nice it would have been if already in those days we had known that this occasion awaited us just thirty years hence.

But it is time to show you that Ernst did teach me some topology there in Princeton. I apologize to have nothing to tell you which would be more apropos to this conference. (There is some irony in that, for in 1949 networks, graphs and such like were my interests. It nearly seems that in our interaction in Princeton, Ernst and I interchanged our momentum.) But before I launch into a subject which I fell in love with, also in that year of 1949, let me close with one serious word. Thank you Ernst for having been a friend and teacher to me, and I expect to most of us here, in the truest sense: for you aways teach as a friend, and what you teach is always more than the subject.

Now to work. I would like to show you a formula, which seems to me the appropriate generalization of the Morse theory to the equivariant situation. Recall then first of all that the "Morse theory" attaches to every "generic" smooth function f, on a compact manifold, M, a polynomial $\mathcal{M}_t(f) = \sum_p t^{\lambda_p}$, where the summation is over the critical points of f and if p is such a critical point, i.e.

$$\left.\frac{\partial f}{\partial x^i}\right|_p = 0 ,$$

then λ_p denotes the number of negative eigenvalues of the matrix

$$H_p f = \left.\frac{\partial^2 f}{\partial x^i \, \partial x^j}\right|_p .$$

The germ "generic" is here meant in the sense that all these second order matrices be nonsingular. This immediately implies that the critical points of f are finite in number and so $\mathcal{M}_t(f)$ is indeed a well defined polynomial. Now the fundamental first result of the Morse theory is that this polynomial has as a lower bound the Poincaré Polynomial of M:

$$P_t(M) = \Sigma\, t^k \dim H^k(M)$$

where $H^k(M)$ denotes the k-th cohomology group of M (relative to some field).

I will not try to define $P_t(M)$ in greater detail here—suffice it to say that it is our oldest and most trustworthy *topological invariant of a space*, and that its coefficients are called the Betti-numbers of M.

Precisely, then the Morse theorem can be stated as follows: *In the situation envisaged there exists a polynomial* $Q(t)$ *with nonnegative coefficients such that*

$$(*) \qquad \mathcal{M}_t(f) - P_t(M) = (1+t)\,Q(t)\,.$$

Note that this certainly implies that $\mathcal{M}_t(f)$ is $\geqslant P_t(M)$ coefficient by coefficient. In fact it is a quite considerable refinement of that inequality. For instance, it immediately implies the following *Lacunary principle* of Morse:

If the product of any two consecutive coefficients of $\mathcal{M}_t(f)$ *vanish, then*

$$\mathcal{M}_t(f) = P_t(M)\,,$$

and M *is "free of torsion".*

For example the 2-coordinate is a function on the 2-sphere: $x^2 + y^2 + z^2 = 1$, with a minimum at $z = 0$ and a maximum at $z = 1$. Hence

$$\mathcal{M}_t(z) = 1 + t^2$$

and so $P_t(S^2) = 1 + t^2$ also.

In short, as a zeroeth approximation to $P_t(M)$, one may use $\mathcal{M}_t(f)$ for an economical f, i.e. one having as few critical points as M will allow. Put differently, we may think of the Morse theory as *quantitatively predicting the minimal critical behavior forced by the topology of the situation.*

The problem I want to address here now, is, how other constraints on a function force critical points, and in particular I would like to measure the effect which an assumed symmetry of f has on its critical behavior.

Two examples will serve to set the stage:

Example 1. Let f be a smooth real valued function on the line **R**, which is invariant under the group of integers, **Z**, acting on **R** by translation:

$$f(x+1) = f(x)\,.$$

What then is the minimal critical behavior of f?

Answer. Clearly such an f descends to a function

$$f_{\mathbf{Z}} : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$$

on the circle obtained by identifying the points of **R** under the group-action, and as such it must satisfy the Morse inequalities of a function on S^1.

Example 2. Let the circle S^1 act on $S^2 \in \mathbf{R}^3$ by rotation about the 2-axis in 3-space. What is the appropriate critical behavior forced on a function f invariant under this action?

If we attempt to treat this case as before by descending to the function

$$f_{S^1} : S^2/S^1 \to \mathbf{R}$$

induced by f, something goes wrong, because S^2/S^1 is clearly just an interval $-1 \leqslant z \leqslant 1$, and hence its Poincaré polynomial is just 1. Thus the Morse inequalities would only force one critical point which is patently absurd.

Now the main difference between these two examples is that in the first one the action of the group was—what we call "free" and in the second it was not. By the way, an action of a group G on a space X is "free" if (roughly) the G-orbit of each point x is homeomorphic to G, and these orbits behave uniformly under the variation of x. This is clearly the case in the first example, and fails near the North and South poles of S^2 in the second.

In our context it is now easily seen that indeed, if G acts freely on M, and f is G invariant then the Morse inequalities of the induced function f_G on M/G express the critical behavior that is forced on f by the symmetry G. Note that in this case M/G will also be a manifold in its own right!

However when the action is not free, M/G will in general not be a manifold at all and its Poincaré series seems to have little to do with the additional critical behavior forced on f by the G-symmetry.

Now in homotopy theory this phenomenon is well-known and the homotopy theorists long ago devised a method of dealing with it. The point is that they defined a *homotopy quotient* for any action of a group G on a space X—notated X_G—which, when the action is free reduces to X/G in the homotopy category, and in other cases produces a new space out of the action. This space X_G reflects the singularities of the action in a—at times—surprising manner. The plan of the homotopy theorists is as follows:

First note that if G acts *freely* on a space W, then the diagonal action of G on $X \times W$ will be *free* whatever the action of G on X was. Secondly, recall that *contractible spaces* do not affect homotopy in any way. In line with these two principles the homotopy quotient X_G is defined as follows: We first select a fixed space W on which G acts satisfying two conditions:

a) G acts freely on W

b) W is contractible.

Then define X_G by:

$$X_G = X \times W/G.$$

Note by the way that according to this procedure the homotopy quotient of the "worst possible of actions" i.e. the action of G on a point p is given by

$$p_G = p \times W/G = W/G.$$

This space—which we call the classifying space of G, and denote by BG is absolutely fundamental in modern homotopy theory. The reason is, that it turns out that a space W satisfying a, b si essentially *uniquely determined by G* (in homotopy theory) so that finally BG is a *space* which depends on G alone and somehow is an amalgam of its own *topology* and its *abstract group structure.*

The following is a table of BG's for some garden variety of groups.

TABLE OF CLASSIFYING SPACES

G	W	W/G	$P_t(W/G)$
$\mathbf{Z}$	$\mathbf{R}$	$\mathbf{R}/\mathbf{Z} = S^1$	$1 + t$
$\mathbf{Z}^n$	$\mathbf{R}^n$	$\mathbf{R}^n/\mathbf{Z}^n = S^1 \times \ldots \times S^1$	$(1+t)^n$
$\mathbf{Z}_2$	$S(H)$	$\mathbf{R}P_\infty$	1
$U(1) = S^1$	$S(H)$	$\mathbf{C}P_\infty$	$(1-t^2)^{-1}$
$U(2)$	2-frames in H	$G_2(H)$	$(1-t^2)^{-1}(1-t^4)^{-1}$
$U(n)$	n frames in H	$G_n(H)$	$(1-t^2)^{-1}\ldots(1-t^{2n})^{-1}$.

Here $\mathbf{Z}$ denotes the integers, $\mathbf{Z}^n$ the direct product of $\mathbf{Z}$ with itself n times, $\mathbf{Z}_2$ the group $\{\pm 1\}$, and $U(n)$ of course the unitary group. The first two W's of course come to mind immediately, on the other hand the rest *should* strike you as *way out.* In all of these think of H as a complex infinite-dimensional Hilbert space, and of $S(H)$ as its unit sphere. The space of n-frames on H is then the space of n-tuples $\{x_1, \ldots, x_n\}$ of elements in $S(H)$ which are mutually orthogonal. $U(n)$ clearly acts on these

$$\{x_i\} \to \{U_{ij}x_j\}$$

and the quotient gives precisely the Grassmannian $G_n(H)$ of all n-dimensional subspaces of H. For $n = 1$, this is simply the projective space.

All these examples then rely on the beautiful fact that the unit sphere in an infinite dimensional Hilbert Space is contractible!

Note also that most of these spaces BG have Poincaré *Series* in accordance with the fact that they are infinite-dimensional manifolds.

We are now finally ready for the equivariant version of the Morse inequalities. Assume then that M and G are compact, that G acts smoothly on M and that f is a G-invariant smooth function on M. The critical points of f then naturally fall into orbits $\{O\}$ of the group action and we now also assume that f is nondegenerate in its category—that is—that the Hessian of f is nondegenerate in the directions *normal to the critical orbits*. It then follows that f has only a finite number of critical orbits $\{O_i\}_{i=1}^n$. Finally let $\{p_i\}_{i=1}^n$ be a set of points one in each orbit O_i, and let H_i be the stability group of p_i. Thus

$$O_i = G/H_i . \quad i = 1, \ldots, n .$$

With all this understood, define the *equivariant* Morse Series of f by:

$$\mathcal{M}_t^G(f) = \Sigma\, t^{\lambda(p)} P_t(BH_p)$$

where p ranges over the $\{p_i\}$.

Also define the equivariant Poincaré series of M by:

$$P_t^G(M) = \Sigma\, t^k \dim H^k(M_G) .$$

(This is then simply the Poincaré series of the homotopy quotient.)

The theorem—due to Atiyah and myself—is now: *For simplicity assume that all the stability groups are connected and both* G *and* M *are compact. Then the equivariant Morse Series of* f *and Poincaré Series of* M *satisfy the Morse inequalities:*

$$(**) \qquad \mathcal{M}_t^G(f) - P_t^G(M) = (1+t)\, Q(t)$$

where $Q(t) = q_0 + q_1 t + \ldots$, *with* $q_i \geqslant 0$.

The proof is really not very difficult, indeed f descends to a function f_G on M_G in the obvious manner and then one simply has to apply the theory of nondegenerate "critical manifolds" to see that

$$\mathcal{M}_t(f_G) = \mathcal{M}_t^G(f) . \qquad \text{Q.E.D.}$$

By the way when one allows for disconnected stability groups, the formula persists with appropriate local coefficient systems. As concrete instance of (**), consider the function

$$f(x, y, z) = z$$

on our two sphere S^2 of Example 2. It is clearly invariant under S^1; Further its critical points are just the North and South poles.—Both are fixed under the action of S^1, thus $H = S^1$ and hence

$$\mathscr{M}_t^G(f) = \frac{1}{1-t^2} + \frac{t^2}{1-t^2}\,.$$

The right-hand side of (**) is also quite computable and one finds that $P_t^G(S^2) = \dfrac{1+t^2}{1-t^2}$ so that the height function z, is again a *perfect* Morse function in the equivariant sense.

Although I am pleased with (**) per se—as I am with all nice formulas—let me just say one word on how M. Atiyah and I chanced upon it, and how we put it to use.

Of late the physicists have been much concerned with the Yang-Mills functional, and its classical extrema. These are then solutions of certain non-linear differential equations, and it occurred to us that the Morse theory should enable us to get at some information about the topology of the space of these solutions. This of course leads to an infinite dimensional analogue of the situation described—but it turns out that in the end it is the formula (**) which leads to the correct result.

For instance applied to the solution space of the Yang-Mills problem—let's call it Min,—for a $U(3)$ bundle over a Riemann Surface of genus g, (**) gives us the following formula for the Poincaré polynomial of Min:

$$(***)\qquad P_t(\mathrm{Min}) = \frac{(t^5+1)^{2g}(t^3+1)^{2g} - (t^2+1)^2 t^{4g-2}(1+t)^{2g}(1+t^3)^{2g} + (1+t^2+t^4)\,t^{6g-2}(1+t)^{4g}}{(t^2-1)\,(t^4-1)^2\,(t^6-1)}\,.$$

I display it proudly to show you that although we are often accused (especially by Ernst) of dealing in chimerial concepts, they do at times solidify into very concrete information.—More concrete than any I have seen here—for that matter. This formula is also noteworthy in that it was *not new*! It was derived by a completely different method—using counting procedures in finite fields going back to C. L. Siegel, and the "Weil-Conjectures". The original impetus for that derivation was done by Harder, [2] who did the $U(2)$ case. The formula (***) occurs in a paper by Ramanan and Usha.

In closing let me bring this formula into the context of our seminar by offering it as a provisional proof of the consistancy of mathematics; to tide us over until you, dear Ernst, or some of your friends devise a more convincing one.

Harvard, March 1980

I append the following bibliography in case someone is interested in following some aspect of the subject. A paper, scheduled for the London Math. Journal, is under preparation by Atiyah and myself.

BIBLIOGRAPHY

1 BOTT R. and H. SAMELSON. Applications of the theory of Morse to symmetric spaces. *Amer. J. of Math. 80* (1968) 964-1029.

2 HARDER G. Eine Bemerkung Zu einer Arbeit von P. E. Newstead. *Jour. für Math. 242* (1970) 16-25.

3 HARDER G. and M. S. NARASIMHAN. On the cohomology groups of moduli spaces of vector bundles over curves. *Math. Ann. 212* (1975) 215-248.

4 MUMFORD D. and P. E. NEWSTEAD. Periods of a moduli space of bundles on curves. *Amer. J. Math. 90* (1968) 1201-1208.

5 NARASIMHAN M. S. and S. RAMANAN. Moduli of vector bundles on a compact Riemann surface. *Ann. of Math. 89* (1969) 19-51.

6 NARASIMHAN M. S. and S. RAMANAN. Vector bundles on curves. *Proceedings of the Bombay Colloquium of Algebraic Geometry* 335-346 Oxford University Press, 1969.

7 NARASIMHAN M. S. and C. S. SESHADRI. Stable and Unitary vector bundles on a compact Riemann surface. *Ann. of Math. 82* (1965) 540-567.

8 NEWSTEAD P. E. Topological properties of some spaces of stable bundles. *Topology 6* (1967) 241-262.

9 —— Stable bundles of rank 2 and odd degree over a curve of genus 2. *Topology 7* (1968) 205-215.

10 —— Characteristic classes of stable bundles of rank 2 over an algebraic curve. *Trans. Amer. Math. Soc. 169* (1972) 337-345.

11 —— Rationality of moduli spaces of stable bundles. *(To appear)*.

12 SESHADRI C. S. Space of unitary vector bundles on a compact Riemann surface. *Ann. of Math. 85* (1967) 303-336.

13 —— Moduli of π-vector bundles over an algebraic curve. *Questions on Algebraic Varieties* Roma 1970.

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