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Artikel: THE FAST SKEW-CLOSURE ALGORITHM
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for i : = 1 step 1 until n do

    begin k : = i + 1 step 1 until (k=n or A_ik=1)
          if A_ik = 1 then A_k* : = A_k* ∨ A_i*
          else for m : = 1 step 1 until i − 1 do
               if A_im = 1 then A_m* : = A_i*
    end
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It is not obvious that this algorithm has such a low time complexity, since it appears that the row copying step may be performed $O(n^2)$ times. However when the correctness of the algorithm is understood it becomes clear that each row is copied *into* at most once and so the total number of these operations is indeed $O(n)$.

We can give an informal proof using Theorem 1 that this algorithm is correct. We may think of A as representing an undirected graph on the index set $\{1, ..., n\}$. Since the algorithm causes no interaction between rows or columns corresponding to different components of the graph, it is sufficient to regard each component separately. We need only prove the correctness for a graph with a single component. It is plain that the nth row is the same after either the original algorithm or after $\Psi\Psi'$. By Theorem 1, this must be all 1's provided that $n > 1$. But the copying operation of the original algorithm must have copied 1's throughout the entire matrix. This is correct.

We shall consider only our refined algorithm in further detail since it has natural generalizations which the old algorithm does not possess.

3. Basic closures

A matrix, A, regarded as a relation, is transitive if $A \geqslant A^2$. The transitive closure of A, A^T, is the least transitive matrix, X, containing A, and we may write

$$A^T = \mu X . X \geqslant A \vee X^2$$

(A^T is often denoted A^+). Similarly for the reflexive closure and symmetric closure

$$A^R = \mu X . X \geqslant A \vee I$$

$$A^S = \mu X . X \geqslant A \vee \overline{X}$$

We have also the reflexive-and-transitive closure

$$A^* = \mu X . X \geqslant A \vee I \vee X^2 .$$

Indeed for any formal polynomial P over $X, \bar{X}$, using product and disjunction with I as the identity, since $A \vee P(X, \bar{X})$ is monotonic in X, we have a unique minimal fixpoint $(\mu X . X \geqslant A \wedge P)$. Our interest in skew-closure is illuminated by the following result:

THEOREM 2.

$$A^{\Omega} = \mu X . X \geqslant A \vee \bar{X} X .$$

Proof. Firstly, A^{Ω} satisfies the inclusion.

$$A \vee \overline{A^{\Omega}} A^{\Omega} = A \vee (\bar{A} \vee \bar{A}(A \vee \bar{A})^* A)(A \vee \bar{A}(\bar{A} \vee A)^* A)$$
$$\leqslant A \vee \bar{A}(\bar{A} \vee A)^* A = A^{\Omega}$$

We note that $\bar{\bar{A}} = A$, $\overline{AB} = \bar{B}\bar{A}$ and $\overline{A^*} = \bar{A}^*$

Secondly, A^{Ω} is minimal. Suppose $A^{\Omega} \underset{\neq}{>} K = (\mu X . X \geqslant A \vee \bar{X} X)$ and let m be the smallest integer such that $Q_m = A \vee \bar{A}(\bar{A} \vee A)^m A$ not $\leqslant K$. Obviously $A \leqslant K$, but also

$$\bar{A}(\bar{A} \wedge A)^m A \leqslant \bigvee_{o \leqslant r < m} \bar{A} A^r . \bar{A}(\bar{A} \vee A)^{m-r-1} A \vee \bar{A} A^m . A$$
$$\leqslant \bigvee_{r<m} \overline{Q_r} . \bigvee_{r<m} Q_r$$
$$\leqslant \bar{K} . K \qquad \text{by minimality of } m$$
$$\leqslant K \qquad \text{by fixpoint property of } K$$

This contradiction proves the Theorem. □

There are just two other monomials in $X, \bar{X}$ of degree at most two, namely $X\bar{X}$ and $\bar{X}\bar{X}$. The first yields a closure, Q', which is merely dual to skew-closure. The second yields a rather curious closure, T', which can be represented by the set of products over $A, \bar{A}$, defined by the strings

$$\{w \in \{A, \bar{A}\}^* \mid \text{number of } A\text{'s} \equiv 1 + \text{number of } \bar{A}\text{'s mod } 3\} \neg\, A(\bar{A}A)^+$$

Since the set of products defining T' is a regular set, this closure is computable using some fixed number of products, transitive closures, disjunctions and transposes. Therefore its computational complexity (like that of product [2]) is no greater than that of T, to within a constant factor. However we have been unable to show the converse.

Open Problem 1. Is there an $O(n^2)$ matrix-based algorithm for the T'-closure?

3. The quadratic monoid

To satisfy our curiosity we investigated the monoid generated by the composition of closures corresponding to polynomials of degree at most two. For any set of transformations E let M_E be the monoid generated by compositions of elements of E. For any polynomial $P(X, \bar{X})$, define $Z_P : A \to (\mu X. X \geqslant A \vee P(X, \bar{X}))$
and then

$$\Pi_r = \{ Z_P \,|\, \deg(P) \leqslant r \}.$$

THEOREM 3. $M_{\Pi_2} = M_{\{R,S,Q,Q',T,T'\}}$ *and the monoid is finite.*

Proof. The equality follows from the finiteness since

$$Z_{P_1 \vee P_2} = \bigvee_m (Z_{P_1} \cdot Z_{P_2})^m$$
$$\in M_{\{Z_{P_1}, Z_{P_2}\}} \quad \text{if this is finite.}$$

$M_{\{R,S,Q,Q',T,T'\}}$ is examined explicitly below and is found to contain exactly fifty elements. □

We write A for the monoid identity given by $A^A = A$ and $[Z_1, ..., Z_k]$ for the closure $\bigvee_m (Z_1 \vee ... \vee Z_k)^m$. Together with the obvious idempotencies of closures we have the following sufficient defining relations.

$$W \stackrel{\text{def}}{=} [S, Q, Q', T, T']$$
$$= QQ' = Q'Q = QT' = Q'T' = SQ = SQ' = ST = ST'$$
$$V \stackrel{\text{def}}{=} [R, S, Q, Q', T, T'] = WR = RQ = RQ' = RT'$$
$$QT = [Q, T] \qquad Q'T = [Q', T]$$
$$T'Q = T'TQ = T'QT \qquad T'Q' = T'TQ' = T'Q'T$$
$$TT' = T'T \;\; * \stackrel{\text{def}}{=} RT = TR \qquad RS = SR$$

The closures in the monoid are

$V \quad : \quad A^V = (\bar{A} \vee A)^*$

$W \quad : \quad A^W = (\bar{A} \vee A)^+$