LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE

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by Steven ZUCKER [1]

INTRODUCTION

It was my original goal, in working on this paper, to show the link between the Hodge theory of vector-valued forms on (compact) quotients of Hermitian symmetric spaces to the Hodge theory for local systems that underlie polarized variations of Hodge structure. The first was the object of study over fifteen years ago, notably in the work of Matsushima, Murakami and Kuga; the second is an unpublished construction of Deligne that has been described and used in my recent work [11]. This paper still fulfills its expository function, and it should serve to unify related ideas, juxtaposing techniques from representation theory and transcendental algebraic geometry. However, it now seems likely that each subject will benefit from techniques and ideas drawn from the other, as attested by the results in Sections 4 and 5. A starting point for this overlap already appears in [11, §12].

We begin the description of the subject matter of this paper. Let $\mathbf{V}$ be a locally constant sheaf on the compact Kähler manifold S, with $\mathbf{V}$ underlying a polarized variation of Hodge structure. One can decompose the $\mathbf{V}$-valued forms on S into components of type $(p, q); (r, s)$: (p, q)-forms with values in the (r, s) Hodge decomposition bundle. [Note that $r + s$ will be equal to the weight of the variation of Hodge structure, and $p + q$ will usually be held fixed (though arbitrary), so there are really only two independent parameters.] Then, according to Deligne, the harmonic forms, and therefore the cohomology $H^{\bullet}(S, \mathbf{V})$, decompose according to "total" bidegree (P, Q), where $P = p + r$ and $Q = q + s$.

Now let G be a real semi-simple Lie group with finite center, K a maximal compact subgroup, Γ a cocompact discrete subgroup of G, $S = \Gamma\backslash G/K$, and

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(ρ, V) a finite-dimensional representation of G. There is a simple construction of a locally constant sheaf $\mathbf{V}$ on S, such that there is a natural isomorphism

$$H^{\bullet}(\Gamma, V) \simeq H^{\bullet}(S, \mathbf{V})$$

If G/K is *Hermitian* symmetric, then according to Matsushima and Murakami [7] the harmonic forms decompose according to (p, q) type. It was this apparent variance with Deligne's result that aroused my interest in this material.

There is a natural "locally homogeneous" (complex) variation of Hodge structure (4.9) on S, determined by the decomposition of V into character spaces under the center of K. The variation is *real* if (ρ, V) is. By combining Deligne's theory and that of [7], we see that there is a complete decomposition of harmonic forms into (p, q); (r, s) components in this case. It is also possible to see this directly from the identities (3.12) among the various Laplacians. In fact, an even finer decomposition is possible (see (3.29)), similar to the one in the main theorem of [2]. We draw algebraic consequences in hypercohomology in (5.16) and (5.23).

Ultimately, one would like to understand the cohomology in the case of *non-compact* locally symmetric varieties S (of finite volume). In most cases, according to [18], this is tantamount to saying that Γ is an arithmetic subgroup of G. The Hodge-theoretic techniques are, in a sense, "formal", and they yield a decomposition theorem for the intrinsic L_2 cohomology $H^{\bullet}_{(2)}(S, \mathbf{V})$ (where exactness conditions must involve only L_2 forms) relative to natural metrics. The precise relation between the L_2 and total cohomology groups is only beginning to emerge (see [12]). The case $G = SL(2, \mathbf{R})$, where S is an algebraic curve, has been treated in [11]. Here, the L_2 cohomology is naturally isomorphic to some "topological" cohomology $H^{\bullet}(\bar{S}, \bar{\mathbf{V}})$ on the smooth compactification $\bar{S}$ of S, where $\bar{\mathbf{V}}$ is a certain extension to $\bar{S}$ of the sheaf $\mathbf{V}$ (see [11, §6]). I suspect that it is possible to describe the L_2 cohomology for arbitrary G in terms of data on a suitable compactification of S (see [12, (3.99)]).

The use of Deligne's Hodge decomposition in the locally homogeneous case permitted us in [11, §12] to arrive at an explanation of the "mysterious" isomorphism of (Eichler-) Shimura. Let $\mathbf{V}$ be $\mathrm{Symm}^k\, \mathbf{C}^2$, as a representation of $SL(2, \mathbf{R})$. The parabolic cohomology $H^1_P(\Gamma, V)$ is naturally isomorphic to $H^1(\bar{S}, \bar{\mathbf{V}})$, and the cusp forms of weight $k + 2$ for Γ determine $\mathbf{V}$-valued holomorphic 1-forms on S that are L_2 in the Poincaré (Bergman) metric. This gives the $(k+1, 0)$-component *(sic)* of the Hodge structure, and one can see directly that its complex conjugate is the only other term which can be non-zero.

In this paper, we prove for arbitrary G the cohomological result (5.29) which underlies the Shimura isomorphism in the case of $SL(2, \mathbf{R})$. It is most easily stated in terms of the holomorphic de Rham complex $\Omega^{\bullet}_S(\mathbf{V})$.The Deligne Hodge filtration (associated to the (P, Q) decomposition) is placed on $\Omega_S^\bullet(\mathbf{V})$, and we are able to determine the cohomology sheaves for the successive quotients as [the locally-free sheaves of sections of] locally homogeneous vector bundles associated to certain representations of K. Of course, $\Omega_S^\bullet(\mathbf{V})$ is itself comprised of homogeneous vector bundles, and the main point is to recognize that each successive quotient corresponds to a single and distinct character under the action of a certain subgroup of the center of K. The representations of K which occur in the cohomology sheaves are none other than the character spaces in the Lie algebra cohomology $H^\bullet(\mathfrak{p}^+, V)$. From this, we are also able to illustrate how, for real representations, cohomology vanishing theorems might be proved by exploiting the fundamental role of the center of K (see (5.34) ff.). However, this method does not seem to produce new results.

From the point of view of the transcendental algebraic geometer, locally homogeneous variations of Hodge structure are interesting as a class of examples of variations of Hodge structure in several variables. In the case of one variable, Schmid's SL_2-orbit theorem [19] can be interpreted as saying that every variation of Hodge structure is asymptotic, near a singularity, to a locally homogeneous one for $SL(2, \mathbf{R})$. This gives a fairly complete description in one variable. However, there is currently no generalization to several variables, nor are the asymptotics of the Hodge norm (a corollary of the SL_2-orbit theorem in the one-variable case) understood. Since the locally homogeneous variations of Hodge structure are so explicit, it should be possible to calculate the asymptotics directly, relative to a nice compactification of S. Hopefully, this will give us a clue to the general situation; it will certainly provide a lower bound for the content of a general theory.

I wish to thank Pierre Deligne for raising questions which extended the scope of this work. I am grateful to David DeGeorge and Nolan Wallach for discussions on Lie groups and symmetric spaces, and to Stephen Greenfield for reading the original manuscript and suggesting improvements. In addition, I would like to thank the referee for proposing improvements in the content and exposition.

TABLE OF CONTENTS

§1. PRELIMINARIES

Let G be a connected real semi-simple Lie group with finite center, K a maximal compact subgroup of G, and let $\mathfrak{g} \supset \mathfrak{k}$ be the corresponding Lie algebras. For any sub-algebra $\mathfrak{a} \subset \mathfrak{g}$, we put

$$\mathfrak{a}_{\mathbf{C}} = \mathfrak{a} \otimes_{\mathbf{R}} \mathbf{C}\,.$$

If B denotes the Killing form of $\mathfrak{g}$, B is negative-definite on $\mathfrak{k}$, and we let $\mathfrak{p}$ denote the orthogonal complement under B of $\mathfrak{k}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a so-called Cartan decomposition of $\mathfrak{g}$, and B is positive-definite on $\mathfrak{p}$.

Let $M = G/K$, the corresponding symmetric space. As $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, B defines an (Ad K)-invariant inner product on $\mathfrak{p}$; and since we may identify $\mathfrak{p}$ naturally as the tangent space to M at the identity coset $x_0 = K$, B determines a unique Riemannian metric on M which is invariant under the canonical left G-action.

Assume initially that M is an *irreducible* symmetric space. Then, if one wishes, G can be taken to be a non-compact almost simple group (i.e., $\mathfrak{g}$ is a simple Lie algebra). In that case, the space M admits a homogeneous complex structure, and becomes an *Hermitian* symmetric space, precisely when $\mathfrak{k}$ has a non-trivial center $\mathfrak{z}$. In this case, dim $\mathfrak{z} = 1$, and $Z = \exp \mathfrak{z}$ is the identity component of the center of K. Let G^{ad} denote the adjoint group of G (i.e., the automorphism group of M) and let K^{ad}, Z^{ad} be the corresponding subgroups of G^{ad}. A choice of $z_0 \in Z^{\mathrm{ad}}$ of order 4 (for which Ad (z_0^2) is a Cartan involution of $\mathfrak{g}$) determines an almost-complex structure on $\mathfrak{p}$:

$$\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

with

$$(1.1)\qquad \begin{cases} \mathfrak{p}^+ = \{X \in \mathfrak{p}_{\mathbf{C}} : \mathrm{Ad}\,(z_0)\,X = iX\} \\ \mathfrak{p}^- = \{X \in \mathfrak{p}_{\mathbf{C}} : \mathrm{Ad}\,(z_0)\,X = -iX\} \end{cases}$$

This determines, via left-translation under G, a Kählerian complex structure on M, such that the action of G is by holomorphic isometries.

For purposes of numeration, we define $\mu = \mu(G)$ to be the degree of the covering map $Z \to Z^{\mathrm{ad}}$. It has the following properties:

(1.2) i) if G is of adjoint type, $\mu = 1$ (cf. [16, (1.17B)]),

ii) if $G' \to G$ is a finite covering, then $\mu(G)$ divides $\mu(G')$,

iii) if $G = SU(n, 1)$, then $\mu = n + 1$.

Let ρ be an irreducible representation of G on the finite-dimensional complex vector space V. We will say that (ρ, V) is a *real* representation if there is a G-invariant $\mathbf{R}$-subspace $V_{\mathbf{R}}$ of V with

$$V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

such that G acts on V by extension of scalars. Under the subgroup Z, the representation necessarily splits into one-dimensional Z-invariant summands, on each of which Z acts by a character. We pick an isomorphism

$$(1.3)\qquad \phi : Z \simeq S^1 = \{w \in \mathbf{C} : |w| = 1\}.$$

The character group of Z is free cyclic, with elements

$$\chi_n : Z \to S^1$$

given by

$$\chi_n(z) = [\phi(z)]^n.$$

Let

$$(1.4)\qquad V\langle n\rangle = \{v \in V : \rho(z)\,v = \chi_n(z)\,v \quad \text{if} \quad z \in Z\},$$

so that

$$(1.5)\qquad V = \oplus_{n\in\mathbf{Z}} V\langle n\rangle.$$

Each $V\langle n\rangle$ is invariant under K. If (τ, W) is a representation of K, then we define $W\langle n\rangle$ as in (1.4); if W is irreducible, then $W = W\langle n\rangle$ for some n, i.e., Z acts by a single character. We will assume to have chosen the isomorphism (1.3) so that Z acts on $\mathfrak{p}^+$ by a "positive" character.

(1.6) *Example.* Assuming that G is almost simple, we take $V_{\mathbf{R}} = \mathfrak{g}$, and $\rho = \mathrm{Ad}$, the adjoint representation of G. Then $\mathfrak{p}^+ = V<\mu>$, $\mathfrak{k}_{\mathbf{C}} = V<0>$, and $\mathfrak{p}^- = V<-\mu>$.

For an irreducible (finite-dimensional) representation of G, we also use ρ to denote the induced action of $\mathfrak{g}$ on V. Because of the above description (1.6) of Ad, it is easy to see that the following hold:

(1.7)
i) $\rho(\mathfrak{p}^+)\, V<n> \subset V<n+\mu>$, $\rho(\mathfrak{k})\, V<n> \subset V<n>$,
$\rho(\mathfrak{p}^-)\, V<n> \subset V<n-\mu>$.

ii) $\{n : V<n> \neq 0\} = \{\lambda, \lambda-\mu, \lambda-2\mu, \ldots, \lambda-m\mu\}$
for some integers $\lambda \geqslant 0$, $m \geqslant \mu^{-1}\lambda$.

iii) If V is real, then for all n,

$$V<-n> = \overline{V<n>} \qquad \text{(complex conjugate)}$$

and thus $m\mu = 2\lambda$.

((1.7 i) includes, in particular, the standard fact that $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are Abelian Lie subalgebras of $\mathfrak{g}_{\mathbf{C}}$.)

For the general case, write

$$G = (\Pi_{j=1}^{l} G_j) \times H, \tag{1.8}$$

where each G_j is almost simple and of non-compact Hermitian type, and H is compact.[1]) Let

$$K = (\Pi_{j=1}^{l} K_j) \times H;$$

$Z = \Pi_{j=1}^{l} Z_j$; and $Z^{\mathrm{ad}} = \Pi_{j=1}^{l} Z_j^{\mathrm{ad}}$, a product of circles. Let Δ^{ad} be the diagonal of Z^{ad}, and Δ the inverse image of Δ^{ad} in Z. One may proceed as before, if we replace Z by Δ. Alternatively, every irreducible representation (ρ, V) of G decomposes as a tensor product

$$(\otimes_{j=1}^{l} (\rho_j, V_j)) \otimes (\sigma, W)$$

in accordance with the product structure (1.8). It is then easy to see that under the action of Δ, the decomposition (1.5) of V is the tensor product of the

[1]) We allow compact factors because of (2.7).

corresponding decompositions of each V_j into character spaces under Z_j, tensored with the "trivial" factor W.

On V there exists a positive-definite Hermitian form (the *admissible inner product*) $T(v, w)$ (see [7, p. 375]), determined uniquely up to a constant multiple, with the property that

$$(1.9) \qquad \text{i) } T(\rho(k)v, \rho(k)w) = T(v, w) \quad \text{if } k \in K$$
$$\text{ii) } T(\rho(X)v, w) = T(v, \rho(X)w) \quad \text{if } X \in \mathfrak{p}.$$

This follows from the fact that $\mathfrak{k} \oplus i\mathfrak{p}$ is a compact Lie algebra. If V is real, then the admissible inner product can be seen to be the Hermitian extension of a real inner product on $V_{\mathbf{R}}$.

Let I denote the intersection of the kernels of all finite-dimensional representations of G. Then I is a central subgroup, and G/I admits the structure of a real (affine) algebraic group. Since we are interested in G only for its finite-dimensional representations and the symmetric space M, we may replace G by G/I and assume that G is an algebraic group. To get all of the representations of $\mathfrak{g}$, it is convenient in the abstract to replace G by its algebraic universal covering group (i.e., one makes the preceding construction for the topological universal cover of G); thus, *we may and do assume that* G *is algebraically simply connected.* We remark that by (1.2), the number $\mu(G)$ can be arbitrarily large, even under this restriction.

Let, then, $G_{\mathbf{C}}$ denote the set of complex points of G. It is a simply-connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Let $K_{\mathbf{C}}$ denote the connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k}_{\mathbf{C}}$. By general theory (see [17, XVII.5]), $K_{\mathbf{C}}$ is the universal complexification of K, and so, by definition, every representation of K extends to a holomorphic representation of $K_{\mathbf{C}}$.

Assume that M is Hermitian, and let P^+ (resp. P^-) denote the subgroup of $G_{\mathbf{C}}$ corresponding to the subalgebra $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) of $\mathfrak{g}_{\mathbf{C}}$. Then $P^+K_{\mathbf{C}}P^-$ is an open subset of $G_{\mathbf{C}}$ which contains G (see [4, p. 317]). Moreover, $G \cap K_{\mathbf{C}}P^- = K$ (see [4, p. 318]), so the mapping of $G \to G_{\mathbf{C}}$ induces a holomorphic embedding

$$(1.10) \qquad M \to \check{M} = G_{\mathbf{C}}/Q;$$

as $Q = K_{\mathbf{C}}P^-$ is a parabolic subgroup of $G_{\mathbf{C}}$, $\check{M}$ is compact and is known as the *compact dual* of M.

§2. Vector bundles on $\Gamma\backslash M$

Let Γ be a discrete subgroup of G which acts freely on the symmetric space M, and put $S = \Gamma\backslash M$. We will discuss two standard constructions of vector bundles on S.

The first type is the quotient by Γ of a homogeneous vector bundle on M. Specifically, let (τ, W) be a finite-dimensional representation of K. Then $E(\tau)$ is defined as the quotient of $G \times W$ by the following identification under the action of K:

$$(2.1) \qquad (g, w) \sim \left(gk^{-1}, \tau(k)\, w\right) \quad \text{if} \quad k \in K\,.$$

$E(\tau)$ is naturally a C^∞ vector bundle on M, and the left action of G on M is covered by the obvious left action of G on $E(\tau)$. Thus, we may take the quotient by any Γ as above to obtain a bundle $E(\Gamma, \tau)$ on S. Alternatively, $E(\Gamma, \tau)$ is an associated vector bundle of the principal K-bundle $\Gamma\backslash G$. Note that if (τ, W) decomposes as a representation of K into

$$(\tau, W) = \oplus_{i=1}^{l} (\tau_i, W_i)\,,$$

then one gets an induced decomposition

$$(2.2) \qquad E(\Gamma, \tau) \simeq \oplus_{i=1}^{l} E(\Gamma, \tau_i)\,.$$

We may identify sections of $E(\Gamma, \tau)$ as the Γ-invariant sections of $E(\tau)$, which in turn are given by mappings $\phi: G \to W$ which satisfy

$$(2.3) \qquad \phi(\gamma g k^{-1}) = \tau(k)\, \phi(g) \quad \text{for all} \quad \gamma \in \Gamma,\ g \in G,\ k \in K\,.$$

An Hermitian metric can be placed on $E(\Gamma, \tau)$ by a choice of $\tau(K)$-invariant inner product on W. (Such exist because K is compact.) The corresponding constant metric on $G \times W$ descends to $E(\Gamma, \tau)$, in view of (2.1).

(2.4) *Example.* Taking $\tau = \mathrm{Ad}\,|_{\mathfrak{p}_{\mathbf{C}}}$, we have a natural isomorphism of $E(\tau)$ and the complexified tangent bundle to M, and we may take quotients by Γ.

The second type of vector bundle is the flat bundle associated to a finite-dimensional representation (ψ, V) of Γ. We let $\Phi(\psi)$ denote the quotient of $M \times V$ under the action of Γ:

$$(m, v) \sim \left(\gamma m, \psi(\gamma)\, v\right).$$

Sections of $\Phi(\psi)$ are given by functions $f: M \to V$ such that

$$f(\gamma x) = \psi(\gamma) f(x) \quad \text{if} \quad \gamma \in \Gamma,\ x \in M. \tag{2.5}$$

The local sections of $\Phi(\psi)$ determined by constant V-valued functions determine a flat structure on $\Phi(\psi)$, whose sheaf of locally constant sections will be denoted $\mathbf{V}$.

The two constructions above are related by the elementary

(2.6) PROPOSITION. *Let* (ρ, V) *be a representation of* G *(which then restricts to representations of* K *and* Γ*). Then the mapping*

$$\tilde{\Xi}: G \times V \to G \times V,$$

defined by $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1} v)$, *induces an isomorphism of* C^∞ *vector bundles*

$$\Xi: \Phi(\rho|_\Gamma) \xrightarrow{\sim} E(\Gamma, \rho|_K).$$

(2.7) *Remark.* Let (ρ, V) be a finite-dimensional representation of G, and (ψ, W) a finite-dimensional *unitary* representation of Γ. We note that by the standard ruse of replacing G by $G' = G \times U(W)$, where $U(W)$ denotes the unitary group of W, $V \otimes W$ becomes a representation space for G', and so the bundle $\Phi(\rho|_\Gamma \otimes \psi)$ falls into the class of bundles covered by (2.6).

A natural metric on $\Phi(\rho|_\Gamma)$ is provided by the admissible inner product T (1.9). For $g \in G$, $v, w \in V$, let (at $gx_0 \in M$)

$$<v, w>_{gx_0} = T\left(\rho(g^{-1}) v, \rho(g^{-1}) w\right). \tag{2.8}$$

Since K acts isometrically with respect to T, it follows that (2.8) is well-defined on $M \times V$; and it is evident that the action of Γ is isometric, so (2.8) descends to $\Phi(\rho|_\Gamma)$. T also determines a metric in $E(\Gamma, \rho|_K)$, and it is clear that the mapping Ξ of (2.6) is then an isometry of bundles.

Assume next that M is Hermitian. Then to every finite-dimensional holomorphic representation (σ, W) of Q is associated a $G_{\mathbf{C}}$-equivariant holomorphic vector bundle $\check{E}(\sigma)$ on $\check{M}$, constructed as in (2.1). By restricting to M and taking the quotient by the action of Γ, we obtain the holomorphic vector bundle $\check{E}(\Gamma, \sigma)$ on S. Q-invariant subspaces of W determine holomorphic sub-bundles of $\check{E}(\Gamma, \sigma)$. Along the same lines as (2.6), we have:

(2.9) PROPOSITION. Let (ρ, V) be a representation of G (which then determines representations of Q and Γ). Then the mapping

$$\tilde{\Xi}: G_{\mathbf{C}} \times V \to G_{\mathbf{C}} \times V,$$

defined by $\tilde{\tilde{\Xi}}(g, v) = (g, \rho(g)^{-1} v)$, induces an isomorphism of holomorphic bundles

$$\Xi : \Phi(\rho|_\Gamma) \xrightarrow{\sim} \check{E}(\Gamma, \rho_{\mathbf{C}}|_Q) .$$

Every representation τ of K determines a holomorphic representation of $K_{\mathbf{C}}$, which then extends to a representation σ_τ of Q by setting σ_τ to be trivial on P^-, since K normalizes P^-. The C^∞ isomorphism $E(\Gamma, \tau) \to \check{E}(\Gamma, \sigma_\tau)$ imparts a holomorphic structure to $E(\Gamma, \tau)$; however, an isomorphism (2.2) need not be holomorphically compatible with (2.9).

(2.10) *Example.* Taking $\tau = \mathrm{Ad}^+ = \mathrm{Ad}\, K|_{\mathfrak{p}^+}$ we obtain a holomorphic isomorphism

$$E(\tau) \simeq \Theta_M \qquad \text{(holomorphic tangent bundle of } M) ,$$

and we may take quotients by Γ. Therefore, since the Killing form gives $(\mathfrak{p}^+)^* \simeq \mathfrak{p}^-$ as a representation of K,

$$E(\Gamma, \Lambda^p \mathrm{Ad}^-) \simeq \Omega^p_S .$$

(Here and elsewhere, we identify a vector bundle with its locally free sheaf of germs of sections.)

There is a relation of the preceding to automorphic forms, coming from the following. Let W be a finite dimensional vector space over $\mathbf{C}$. Then an *automorphy factor* $\mathscr{J}$ is a C^∞ mapping

$$\mathscr{J} : G \times M \to GL(W)$$

which satisfies

(2.11) i) $\mathscr{J}(g, x)$ is, for fixed g, a holomorphic mapping from M into $GL(W)$,
ii) $\mathscr{J}(gh, x) = \mathscr{J}(g, hx)\, \mathscr{J}(h, x)$.

We observe that $\mathscr{J}$ is then completely determined by the function $\mathscr{J}(g, x_0)$ on G. Given such a $\mathscr{J}$, one forms the *automorphic vector bundle* $A(\Gamma, \mathscr{J})$, a holomorphic bundle, by taking the quotient of $M \times W$ under the action of Γ:

$$(x, w) \sim (\gamma x, \mathscr{J}(\gamma, x)\, w) \qquad \text{for all} \qquad \gamma \in \Gamma,\ x \in M,\ w \in W .$$

Sections of $A(\Gamma, \mathscr{J})$ are then given by functions $f : M \to W$ such that

(2.12) $$f(\gamma x) = \mathscr{J}(\gamma, x)\, f(x) \qquad \text{for all} \qquad \gamma \in \Gamma,\ x \in M ;$$

these are called *automorphic forms.*

From an automorphy factor j, one obtains a representation τ_j of K by setting

$$\tau_j(k) = j(k, x_0),$$

because of (2.11, ii). We then have

(2.13) PROPOSITION. *Let j be an automorphy factor. Then there is a C^∞ isomorphism*

$$\Psi : E(\Gamma, \tau_j) \to A(\Gamma j),$$

induced by the mapping

$$\tilde{\Psi} : G \times W \to G \times W$$

$$\tilde{\Psi}(g, w) = (g, j(g, x_0)\, w).$$

(2.14) *Remark.* For a representation (ρ, V) of G,

$$j(g, x) = \rho(g)$$

defines an automorphy factor, for which (2.13) is a reformulation of (2.6).

Conversely, to the Lie group G is associated its *canonical automorphy factor* $\mathcal{J}$ (see [7, p. 397]), which is a C^∞ mapping $\mathcal{J} : G \times M \to K_{\mathbf{C}}$ which satisfies the equations of (2.11); and $\mathcal{J}(g, x_0)$ is the $K_{\mathbf{C}}$-component of g in $G \subset U = P^+ K_{\mathbf{C}} P^-$. Then each representation τ of K determines an automorphy factor

$$j_\tau(g, x) = \tau(\mathcal{J}(g, x)).$$

In this case, the mapping $\tilde{\Psi}$ of (2.13) extends to a biholomorphic mapping of $U \times W$, from which it follows that

$$\Psi : E(\Gamma, \tau) \to A(\Gamma, j_\tau)$$

is an isomorphism of holomorphic bundles. Thus we have also, for instance,

$$\Omega^p_S \simeq A(\Gamma, j_{\Lambda^p \mathrm{Ad}^-}).$$

In this manner, holomorphic sections of bundles $E(\Gamma, \tau)$ become given as spaces of automorphic forms. One also uses (2.13) to construct local frames for $E(\Gamma, \tau)$.

§3. The cohomology groups $H^n(\Gamma; \rho, V)$

In this section, we will discuss the various approaches toward computing the Eilenberg-MacLane cohomology groups $H^n(\Gamma; \rho, V)$ for a finite-dimensional representation (ρ, V) of G, which we may as well take to be irreducible.

We begin with the use of deRham cohomology, as carried out originally in [7]. Since M is contractible, there is a natural isomorphism

$$H^n(\Gamma; \rho, V) \simeq H^n(S, \mathbf{V})$$

(with notation as in §2), hence we may compute these cohomology groups from the complex of $\mathbf{V}$-valued C^∞ forms on S (by the deRham theorem).

We will make use of the following obvious diagram of manifolds

$$\begin{array}{ccc} G & \xrightarrow{\psi} & \Gamma\backslash G \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\lambda} \\ M & \xrightarrow{\pi} & S \end{array} \tag{3.1}$$

Let η be an element of $\mathscr{A}^n(S, \mathbf{V})$, the space of global C^∞ n-forms on M with values in $\mathbf{V}$. Then

$$\phi = \kappa^*\pi^*\eta$$

is a V-valued form on G satisfying the equations

(3.2)

$$\text{i) } \gamma^*\phi = \rho(\gamma)\phi \qquad \text{if } \gamma \in \Gamma$$

$$\text{ii) } \mathscr{L}_Y\phi = 0 \qquad \text{if } Y \in \mathfrak{k},$$

$$\mathscr{L}_Y = \text{Lie derivative} = (\Lambda^n \mathrm{Ad}^*)(Y)$$

$$\text{iii) } \iota_Y\phi = 0 \qquad \text{if } Y \in \mathfrak{k}$$

$$\iota_Y = \text{interior multiplication by } Y$$

Conversely, every element $\phi \in \mathscr{A}^n(G) \otimes_{\mathbf{C}} V$ ($\mathscr{A}^n(G)$ denoting the space of C^∞ n-forms on G) that satisfies (3.2) is $\kappa^*\pi^*\eta$ for some $\eta \in \mathscr{A}^n(S, \mathbf{V})$. We then apply the mapping $\tilde{\Xi}$ of (2.6) to ϕ, obtaining the n-form

$$\tilde{\eta} = \rho(g^{-1})\phi \tag{3.3}$$

which satisfies

(3.4) i) $\gamma^*\tilde{\eta} = \tilde{\eta}$ if $\gamma \in \Gamma$,

ii) $\mathscr{L}_Y\tilde{\eta} = -\rho(Y)\,\tilde{\eta}$ if $Y \in \mathfrak{k}$,

iii) $\iota_Y\tilde{\eta} = 0$ if $Y \in \mathfrak{k}$.

In particular, we may view $\tilde{\eta}$ as a vector-valued form on $\Gamma\backslash G$.

We next describe the Hodge theory for $H^n(S, \mathbf{V})$ from this point of view, as was done in [7] and [8]. Actually, one must work with the L_2 cohomology when S is non-compact. Since we have defined a metric on $A(\Gamma, \rho)$ in Section 2, and on the tangent bundle by the Killing form, there is an L_2 norm $\| \eta \|_{(2)}$ for $\eta \in \mathscr{A}^n(S, \mathbf{V})$, and the L_2 cohomology is defined by

(3.5)
$$H^n_{(2)}(S, \mathbf{V}) = \frac{\{\eta \in \mathscr{A}^n(S, \mathbf{V}) : \ \eta \text{ is } L_2 \text{ and } d\eta = 0\}}{\{\eta \text{ as above}: \ \eta = d\psi \text{ for some } L_2 \ \psi \in \mathscr{A}^{n-1}(S, \mathbf{V})\}}$$

There is then an obvious mapping

(3.6)
$$H^n_{(2)}(S, \mathbf{V}) \to H^n(S, \mathbf{V}),$$

and one is ultimately interested in understanding the kernel and image of this mapping. (See also [12].)

(3.7) *Remark.* We may compute the L_2 cohomology groups (3.5) from the complex of weakly differentiable L_2 forms $\mathscr{L}^\bullet_{(2)}(S, \mathbf{V})$; i.e., we may drop the smoothness condition on forms (see [15, §8]). Then d becomes a densely-defined differential for the "complex" of Hilbert spaces of $\mathbf{V}$-valued L_2 forms, and

$$H^n_{(2)}(S, \mathbf{V}) \simeq \frac{\{\text{weakly closed } \mathbf{V}\text{-valued } n\text{-forms}\}}{\{\text{range of } d \text{ on } L_2\ (n-1)\text{-forms}\}}.$$

We define the *reduced* L_2 cohomology $\bar{H}^n_{(2)}(S, \mathbf{V})$ by replacing the range of d in the above quotient by its Hilbert space closure; the reduced L_2 cohomology inherits a Hilbert space structure from the L_2 inner product.

In discussing $\| \eta \|_{(2)}$, we wish to make use of the form $\tilde{\eta}$ of (3.4), and we have

(3.8) LEMMA [7, p. 380]. If $\eta \in \mathscr{A}^n(S, \mathbf{V})$ and $\tilde{\eta} \in \mathscr{A}^n(\Gamma\backslash G) \otimes V$ is the corresponding element, then

$$\| \eta \|^2_{(2)} = c \, \| \tilde{\eta} \|^2_{(2)},$$

where c equals the volume of K.

While much of what follows holds in the absence of a complex structure, we restrict ourselves to the Hermitian symmetric case for the purposes of this exposition. For the general case see [7].

Choose an orthonormal basis $\{X_i\}_{i=1}^k$ of $\mathfrak{p}^+$, so

$$\{X_1, \bar{X}_1, \ldots, X_k, \bar{X}_k\}$$

forms an orthonormal basis of $\mathfrak{p}_{\mathbf{C}}$. For $\eta \in \mathscr{A}^{p,q}(S, \mathbf{V})$, put

$$\eta_{i_1, \ldots, i_p; j_1, \ldots, j_q} = \tilde{\eta}(X_{i_1}, \ldots, X_{i_p}, \bar{X}_{j_1}, \ldots, \bar{X}_{j_q}) \in \mathscr{A}^0(G) \otimes V .$$

Let

$$d = d' + d''$$

be the usual decomposition of the (flat) exterior derivative d on $\mathscr{A}^\bullet(S, \mathbf{V})$ into components of bidegree (1, 0) and (0, 1). The bidegree (1, 0) differential operators D' and d'_p are defined by the formulas

(3.9)
$$(D'\eta)_{i_1, \ldots, i_{p+1}; j_1, \ldots, j_q} = \sum_{u=1}^{p+1} (-1)^{u-1} X_{i_u} \eta_{i_1, \ldots, \hat{i_u}, \ldots, i_{p+1}; j_1, \ldots, j_q} ,$$

(3.10)
$$(d'_\rho\eta)_{i_1, \ldots, i_{p+1}; j_1, \ldots, j_q} = \sum_{u=1}^{p+1} (-1)^{u-1} \rho(X_{i_u}) \eta_{i_1, \ldots, \hat{i_u}, \ldots, i_{p+1}; j_1, \ldots, j_q} .$$

One also puts $D'' = \overline{D'}$ and $d''_\rho = \overline{d'_\rho}$. Then $d' = D' + d'_\rho$ and $d'' = D'' + d''_\rho$; if we put $D = D' + D''$ and $d_\rho = d'_\rho + d''_\rho$, then $d = D + d_\rho$. We remark that D gives a metric connection on $\Phi(\rho)$; heuristically, we regard $\kappa^*E(\rho)$ as being canonically flat.

Let $\mathfrak{D}$ represent any of the above operators. One can obtain directly formulas for the L_2 adjoint $\mathfrak{D}^*$ and the Laplacian

(3.11)
$$\square_{\mathfrak{D}} = \mathfrak{D}\mathfrak{D}^* + \mathfrak{D}^*\mathfrak{D}$$

(see [9, pp. 68-70]). From these calculations, one obtains also the following identities

(3.12) PROPOSITION. *As operators on* $\mathscr{A}^\bullet(S, \mathbf{V})$,

i) $\square_d = \square_{d'} + \square_{d''}$

ii) $\square_d = \square_D + \square_{d_\rho}$

iii) $\square_D = \square_{D'} + \square_{D''}$

iv) $\square_{d_\rho} = \square_{d'_\rho} + \square_{d''_\rho}$

v) $\square_{d'} = \square_{D'} + \square_{d'_\rho}$

(3.13) *Remark.* One always has

$$\square_{(\mathfrak{D}_1+\mathfrak{D}_2)} = \square_{\mathfrak{D}_1} + \square_{\mathfrak{D}_2} + (\mathfrak{D}_1\mathfrak{D}_2^* + \mathfrak{D}_2^*\mathfrak{D}_1 + \mathfrak{D}_1^*\mathfrak{D}_2 + \mathfrak{D}_2\mathfrak{D}_1^*),$$

so (3.12) amounts to establishing the vanishing of the expression in parentheses on the right-hand side. The identities in (3.12) are *not* general formulas for flat bundles on manifolds, but are particular to the group-theoretic context.

Since S is complete in the induced metric from M, the operators $\mathfrak{D}$ as above have unique [3] closed extensions to $\mathscr{L}^\bullet_{(2)}(S, \mathbf{V})$, so the identities (3.12) continue to remain valid in the strict sense on L_2. From this, one may conclude

(3.14) PROPOSITION. *If* $\eta \in \mathscr{L}^\bullet_{(2)}(S, \mathbf{V})$, *the following are equivalent:*

i) $\square_d\eta = 0$ *(η is harmonic),*

ii) $\square_{d'}\eta = \square_{d''}\eta = 0$

iii) $\square_{D'}\eta = \square_{D''}\eta = \square_{d'_\rho}\eta = \square_{d''_\rho}\eta = 0,$

iv) $D'\eta = (D')^*\eta = D''\eta = (D'')^*\eta = d'_\rho\eta$

$= (d'_\rho)^*\eta = d''_\rho\eta = (d''_\rho)^*\eta = 0\,.$

Since $\square_{\mathfrak{D}}$ is elliptic for any of the operators $\mathfrak{D}$ above, harmonic forms are necessarily C^∞. Let $\mathscr{h}^n_{(2)}(S, \mathbf{V})$ denote the space of L_2 harmonic n-forms with values in $\mathbf{V}$. We obtain by standard theory (see [15, §1]):

(3.15) PROPOSITION. *For all* n,

i) $\bar{H}^n_{(2)}(S, \mathbf{V}) \simeq \mathscr{h}^n_{(2)}(S, \mathbf{V})$,

ii) *The mapping* $\mathscr{h}^n_{(2)}(S, \mathbf{V}) \to H^n_{(2)}(S, \mathbf{V})$ *is injective, and is an isomorphism if and only if* d, *operating on* $\mathscr{L}^{n-1}_{(2)}(S, \mathbf{V})$, *has closed range.*

(3.16) *Remark.* An easy way to guarantee that the mapping in (3.15, ii) is an isomorphism is by showing that $H^n_{(2)}(S, \mathbf{V})$ is finite-dimensional.

By (3.14, ii) a form is harmonic if and only if it is annihilated by the Laplacians of the bidegree-preserving operators d' and d''. Therefore, a form is harmonic if and only if its (p, q) components are harmonic, so

$$\mathscr{h}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{p+q=n} \mathscr{h}^{p,q}_{(2)}(S, \mathbf{V})\,. \tag{3.17}$$

Passing this through the isomorphism (3.15, i), we get

$$\bar{H}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{p+q=n} H^{p,q}_{(2)}(S, \mathbf{V})\,. \tag{3.18}$$

If we take S to be compact, we have $H^n_{(2)}(S, \mathbf{V}) = H^n(S, \mathbf{V})$, and in (3.18) the Hodge decomposition of [7].

The most significant assertion about Laplacians, as we will see in Section 5, is given by

(3.19) PROPOSITION [8, p. 14].

$$\square_{D''} + \square_{d'_\rho} = \square_{D'} + \square_{d''_\rho}\,.$$

This fact was not fully exploited in the earlier work.

(3.20) COROLLARY. η is harmonic if and only if

$$\square_{D''}\eta = \square_{d'_\rho}\eta = 0\,.$$

We close this section with a brief account of another way of viewing the cohomology groups $H^n(\Gamma; \rho, V)$, currently preferred in representation theory. For simplicity, we assume that S is compact, and mention at the end what changes must be made in the non-compact case.

From the description (3.4), it is clear that we may regard an element of $\mathscr{A}^n(S, \mathbf{V})$ as a mapping from $\Lambda^n\mathfrak{p}_{\mathbf{C}}$ into $\mathscr{A}^0(\Gamma\backslash G) \otimes V$ that satisfies a transformation rule under $\mathfrak{k}$. This correspondence gives an isomorphism of $H^n(S, \mathbf{V})$ with the *relative Lie algebra cohomology* (see, e.g. [8, pp. 6-8] or [14, Ch. I]):

$$H^n(\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}}, \mathscr{A}^0(\Gamma\backslash G) \otimes V)\,, \tag{3.21}$$

associated to the cochain complex

$$\mathrm{Hom}_K\,(\Lambda^{\bullet}\mathfrak{p}\,\mathscr{A}^0\,(\Gamma\backslash G) \otimes V)\,. \tag{3.22}$$

Here, $\mathfrak{g}_{\mathbf{C}}$ acts on $\mathscr{A}^0\,(\Gamma\backslash G)$ by differentiation, induced by the regular representation of G.

(3.23) *Remark.* By a theorem of van Est (see [5, p. 386]), the relative Lie algebra cohomology is in turn isomorphic to the differentiable (or even continuous) Eilenberg-MacLane cohomology

$$H^n_d\,(G, \mathscr{A}^0\,(\Gamma\backslash G) \otimes V)\,.$$

For this reason, (3.21) is often referred to as "continuous cohomology."

The cohomology (3.21) decomposes according to the splitting of $\mathscr{A}^0\,(\Gamma\backslash G) \otimes V$. First, one decomposes $L_2\,(\Gamma\backslash G)$ as a representation of G:

$$L_2\,(\Gamma\backslash G) \simeq \widehat{\bigoplus_{\alpha}}\, E_\alpha \tag{3.24}$$

into the direct sum of irreducible unitary representations of finite multiplicity. Then

$$L_2\,(\Gamma\backslash G, V) \simeq \widehat{\bigoplus_{\alpha}}\,(E_\alpha \otimes V) \tag{3.25}$$

Taking C^∞ vectors gives the decomposition

$$\mathscr{A}^0\,(\Gamma\backslash G) \otimes V \simeq \widehat{\bigoplus_{\alpha}}\,(E^\infty_\alpha \otimes V)\,, \tag{3.26}$$

By a formula of Kuga (see [7, p. 385] or [14, p. 49]), in terms of the form $\tilde{\eta}$, the Laplacian is given by

$$\widetilde{\Box\eta} = [-C + \rho\,(C)]\,\tilde{\eta}\,, \tag{3.27}$$

where C is the Casimir element of the enveloping algebra of $\mathfrak{g}$. It follows that in each summand of (3.26), there can be non-zero harmonic forms only if the infinitesimal characters χ_α of (π_α, E_α) and χ_ρ of (ρ, V) agree on C. In fact, if the space of harmonic forms is non-zero one must have $\chi_\alpha = \chi_\rho$ (see [1, (2.4)]). In this case, every cochain with values in E_α is harmonic. Thus,

$$\begin{aligned} H^n\,(S, \mathbf{V}) &\simeq \bigoplus_{\chi_\alpha = \chi_\rho} \mathrm{Hom}_K\,(\Lambda^n\mathfrak{p}_{\mathbf{C}}, E_\alpha \otimes V) \\ &\simeq \bigoplus_{\chi_\alpha = \chi_\rho} (\Lambda^n\mathfrak{p}^*_{\mathbf{C}} \otimes E_\alpha \otimes V)^K \qquad (K\text{-invariants}). \end{aligned} \tag{3.28}$$

From (3.27) and (3.28), one obtains the following:

(3.29) PROPOSITION. *Let* (ρ_1, V_1) *and* (ρ_2, V_2) *be two irreducible representations of* G, *and suppose that* $\rho_1(C) = \rho_2(C)$. *Then every morphism of* K-*representations*

$$\phi : \Lambda^{n_1} \mathfrak{p}^* \otimes V_1 \to \Lambda^{n_2} \mathfrak{p}^* \otimes V_2$$

induces a mapping of harmonic forms

$$\phi_* : \mathscr{H}^{n_1}(S, \mathbf{V}_1) \to \mathscr{H}^{n_2}(S, \mathbf{V}_2).$$

and thus a mapping $\phi_* : H^{n_1}(S, \mathbf{V}_1) \to H^{n_2}(S, \mathbf{V}_2)$. *(If the infinitesimal characters of* (ρ_1, V_1) *and* (ρ_2, V_2) *differ, then* ϕ_* *is the zero mapping.)*

If we now decompose each $\Lambda^n \mathfrak{p}_{\mathbf{C}}^* \otimes E_\alpha \otimes V$ as a representation of K and apply (3.29) to the projections onto each component, there is induced decomposition of $H^n(S, \mathbf{V})$, much in the spirit of [2]. If we decompose only $\Lambda^n \mathfrak{p}^*$, we obtain the decomposition (3.18). We will refine that decomposition in §5.

If S is non-compact, then $L_2(\Gamma\backslash G)$ is the direct sum of its discrete spectrum $L_2(\Gamma\backslash G)_d$ and the continuous spectrum $L_2(\Gamma\backslash G)_{ct}$. One then has a decomposition like (3.24) only for $L_2(\Gamma\backslash G)_d$. From there, one obtains an injection

$$\widehat{\bigoplus_\alpha}{}' (E_\alpha^\infty \otimes V) \to \mathscr{A}^0_{(2)}(\Gamma\backslash G) \otimes V, \tag{3.30}$$

whose image consists of those C^∞ $\mathbf{V}$-valued functions for which all left-invariant differential operators are in L_2. Borel has shown that (3.30) induces an isomorphism on cohomology. Also, if Γ is an arithmetic subgroup of G, then all harmonic forms come from $L_2(\Gamma\backslash G)_d$. In this case, one therefore obtains, as in (3.28), the isomorphism

$$\bar{H}^n_{(2)}(S, \mathbf{V}) \simeq \bigoplus_{\chi_\alpha = \chi_\rho} (\Lambda^n \mathfrak{p}_{\mathbf{C}}^* \otimes E_\alpha \otimes V)^K. \tag{3.31}$$

Moreover, the above sum has only finitely many non-zero terms, as the reduced L_2 cohomology is finite-dimensional. Borel discovered the initially surprising phenomenon that the (non-reduced) L_2 cohomology is for some groups infinite-dimensional, with d having non-closed range on the continuous spectrum in certain dimensions; however, this never occurs in the Hermitian case. As a reference for this paragraph, see [13] and the references cited therein [1]. (See also [12] for a different approach to the L_2 cohomology.)

[1]) See note added in proof.

§4. The variation of Hodge structure associated to (ρ, V)

Under the assumption that V is a real representation of G, the Hodge theory for $H^n(S, \mathbf{V})$ fits, perhaps surprisingly, into the more general framework of [11]. Associated to the irreducible representation (ρ, V) of G, there is a homogeneous variation of Hodge structure on the Hermitian symmetric space M, which we shall now define and analyze. The notion turns out to be a variant of ideas in [16, §1].

We recall some basic definitions from Hodge theory. We will use some of the same symbols that were employed in the preceding sections, in a more general context, so that the passage from representation theory to Hodge theory will be clearly laid out. Thus, let $V_{\mathbf{R}}$ be a finite-dimensional real vector space, V its complexification.

(4.1) *Definition.* A *Hodge structure of weight* m on V is a direct sum decomposition

$$V = \bigoplus_{\substack{p,\, q \in \mathbf{Z} \\ p+q=m}} H^{p,q}\ .$$

such that $\overline{H^{p,q}} = H^{q,p}$.

Let $\mathfrak{C}$ be the *Weil operator* of the Hodge structure, defined as the direct sum of the scalar operators i^{p-q} on $H^{p,q}$.

(4.2) *Definition.* A *polarization* of a Hodge structure is a bilinear form $\beta(v, w)$ on $V_{\mathbf{R}}$ such that $\beta(\mathfrak{C}v, \bar{v}) > 0$ whenever $v \neq 0$, and the $H^{p,q}$ spaces are orthogonal with respect to the Hermitian extension of β to V.

(4.3) *Remark.* The definition (4.2) implies the condition usually imposed on β: that it be symmetric if m is even, skew if m is odd.

Ordinarily, the primary example of a polarized Hodge structure of weight m is the cohomology group $H^m(M, \mathbf{C})$ of a compact Kähler manifold M, with a polarization built from cup-product.

(4.4) *Definition.* The *Hodge filtration*

$$\ldots \supset F^r \supset F^{r+1} \supset \ldots$$

is defined by $F^r = \bigoplus_{p \geq r} H^{p,q}$.

One recovers $H^{p,q}$ as $F^p \cap \overline{F^q}$.

Let S be a complex manifold. We will give the definition of a *polarizable* (real) *variation of Hodge structure of weight* m in terms of the universal covering $\pi: M \to S$. Let $\Gamma = \pi_1(S)$, viewed as the group of deck transformations of M. Let $(\rho, V_{\mathbf{R}})$ be a finite dimensional representation of Γ (the *monodromy*). Take $M \times V$: if we place the usual topology on V, we have a vector bundle $\mathscr{V}$; placing the discrete topology on V, we get a constant sheaf $\mathbf{V}$. We require:

(4.5) i) For each $x \in M$, there is a Hodge structure of weight m:

$$V = \bigoplus H_x^{p,q}.$$

ii) For each (p, q),

$$\mathscr{H}^{p,q} = \coprod_{x \in M} (\{x\} \times H_x^{p,q})$$

forms a C^∞ sub-bundle of $\mathscr{V}$.

iii) For each r,

$$\mathscr{F}^r = \coprod_{x \in M} (\{x\} \times F_x^r) \qquad (F_x^r = \bigoplus_{p \geq r} H_x^{p,q})$$

forms a holomorphic sub-bundle of $\mathscr{V}$.

iv) If σ is a local holomorphic section of $\mathscr{F}^r$, and X is a local holomorphic vectorfield on M, then $X\sigma$ is a section of $\mathscr{F}^{r-1}$.

v) There exists a *flat* bilinear form which polarizes the Hodge structure for each x.

(In order to pass this data down to S, we add)

vi) If $\gamma \in \Gamma$, $\rho(\gamma) H_x^{p,q} = H_{\gamma x}^{p,q}$.

We can then take quotients of $\mathscr{V}$, $\mathscr{F}^r$ and $\mathbf{V}$ by Γ to obtain objects on S, which will also be denoted $\mathscr{V}$, $\mathscr{F}^r$ and $\mathbf{V}$, hopefully without confusion.

Ordinarily, the primary example of such $\mathbf{V}$ underlying polarized variations of Hodge structure are the systems $R^m f_* \mathbf{C}$ of m-dimensional cohomology along the fibers for families of Kähler manifolds $f: \mathfrak{M} \to S$.

It is useful to relax the conditions of (4.5). Following a suggestion of P. Deligne, we give:

(4.6) *Definition.* Let (ρ, V) be a finite dimensional representation of Γ (not necessarily real) and $\mathbf{V}$ the resulting locally constant sheaf. By a *complex variation of Hodge structure of weight* m, we mean the collection of data described in (4.5) with the following modifications:

a) For (i), we drop the requirement that $\overline{H_x^{p,q}} = H_x^{q,p}$, for it may be inconsistent with (vi). (One might call the resulting decomposition in (4.1) a *complex Hodge structure.*)

b) To (iii), we add for each s,

$$\bar{\mathscr{F}}^s = \coprod_{x \in M} (\{x\} \times \bar{F}_x^s) \qquad (\bar{F}_x^s = \bigoplus_{q \geqslant s} H_x^{p,q})$$

forms an anti-holomorphic sub-bundle of $\mathscr{V}$.

c) To (iv), add: if σ is a local anti-holomorphic section of $\bar{\mathscr{F}}^r$ and $\bar{X}$ is a local anti-holomorphic vector field on M, then $\bar{X}\sigma$ is a section of $\bar{\mathscr{F}}^{r-1}$.

d) Replace (v) by: there is a flat sequilinear pairing

$$\bar{\beta} : \mathbf{V} \times \mathbf{V} \to \mathbf{C},$$

such that $\bar{\beta}(\mathfrak{C}v, v) > 0$ whenever $v \neq 0$.

(4.7) *Remark.* When V is real, and one has a (real) variation of Hodge structure, then (b) and (c) of (4.6) are automatic, and $\bar{\beta}$ is given by

$$\bar{\beta}(v, w) = \beta(v, \bar{w}).$$

Let now (ρ, V) be an irreducible representation of the Lie group G. Then we may write, according to (1.5) and (1.7)

$$V = \bigoplus_{s=0}^{m} V\langle\lambda - s\mu\rangle, \tag{4.8}$$

where χ_λ is the highest character occurring in (ρ, V). We convert (4.8) into a complex variation of Hodge structure of weight m on $S = \Gamma \backslash M$ by first setting

$$H_0^{p,q} = V\langle\lambda - p\mu\rangle \tag{4.9}$$

if $p, q \geqslant 0$ and $p + q = m$; we then define

$$H_x^{p,q} = \rho(g) H_0^{p,q} \quad \text{if} \quad gx_0 \in M. \tag{4.10}$$

For obvious reasons, we will call this a *locally homogeneous* variation of Hodge structure. It is real whenever (ρ, V) is.

We must verify that the conditions of (4.6) hold; we follow the numbering in (4.5). Because of (1.7, i), the space $H_x^{p,q}$ is well-defined (in the real case, use also (1.7, iii)), so (i) is satisfied. Properties (ii) and (vi) pose no difficulty. We get (iii) from the fact that $F_0^r = \bigoplus_{p \geqslant r} H_0^{p,q}$ (resp. $\bar{F}_0^s = \bigoplus_{q \geqslant s} H_0^{p,q}$) is a $K_{\mathbf{C}}P^-$ (resp.

$P^+K_{\mathbf{C}}$)-invariant subspace of V, and (iv) from the fact that $\mathfrak{p}^+F_0^r \subset F_0^{r-1}$ (resp. $\mathfrak{p}^-\bar{F}_0^s \subset \bar{F}_0^{s-1}$); both assertions follow from (1.7, i) and (4.9).

The flat polarization (4.6, d) ((4.5, v) in the real case) is provided by the admissible inner product T (1.9). Let $\mathfrak{C}_0$ denote the Weil operator of (4.8). Then

$$T(v, w) = \bar{\beta}(\mathfrak{C}_0 v, w) \qquad \text{if we put} \qquad \bar{\beta}(v, w) = T(\mathfrak{C}_0^{-1} v, w)\,. \tag{4.11}$$

We assert that $\bar{\beta}$ is G-invariant (with G acting by $\bar{\rho}$ on the second entry). For this, we need only apply (1.9) to see that

$$\bar{\beta}(\rho(X)\, v, w) + \bar{\beta}(v, \overline{\rho(X)}\, w) = 0$$

for all $X \in \mathfrak{g}_{\mathbf{C}}$, $v, w \in V$. (In the real case, we are displaying the self-contragredience of ρ.) That $\bar{\beta}$ determines a polarization now follows by homogeneity. This completes our verification.

Note that at $gx_0 \in M$,

$$\begin{aligned} \bar{\beta}(\mathfrak{C}_{gx_0} v, w) &= \bar{\beta}(\rho(g)\, \mathfrak{C}_0\, \rho(g)^{-1}\, v, w) \\ &= \bar{\beta}(\mathfrak{C}_0\, \rho(g)^{-1}\, v, \overline{\rho(g)^{-1}}\, w) \\ &= T(\rho(g)^{-1}\, v, \overline{\rho(g)^{-1}}\, w)\,, \end{aligned} \tag{4.12}$$

so the "Hodge metric" coincides with the one given in (2.8). Also, $\eta \in \mathscr{A}^n(S, \mathbf{V})$ takes its values in $\mathscr{H}^{p,q}$ if and only if $\tilde{\eta} \in \mathscr{A}^n(\Gamma\backslash G) \otimes H_0^{p,q}$.

§5. Hodge theory for $H^n(\Gamma; \rho, V)$, from the variation of Hodge structure

In this Section, we will review the general Hodge theory for locally constant sheaves $\mathbf{V}$ underlying polarizable variations of Hodge structure. After that, we will insert the construction of (4.9) into the general framework and draw special conclusions about this case. There are both local considerations and global results. The latter follow "automatically" only when S is compact, in which case they are due to Deligne (see [11, §§1-2]). The global results generalize to non-compact quotients of finite volume for $G = SL(2, \mathbf{R})$ [11, §§7, 12], and hopefully we will soon be able to handle $G = SU(n, 1)$. We should view the compact case as providing formal guidelines for a general theory.

Let **V** underlie a complex polarizable variation of Hodge structure of weight m on the compact Kähler manifold S, as in (4.6). Let

$$\mathscr{A}^{p,q}_{(2)}(S, \mathscr{H}^{r,s})$$

denote the space of square-summable C^∞ forms on S of type (p, q) with values in $\mathscr{H}^{r,s}$. Then

$$\mathscr{A}^{n}_{(2)}(S, \mathbf{V}) = \bigoplus_{\substack{p+q=n \\ r+s=m}} \mathscr{A}^{p,q}_{(2)}(S, \mathscr{H}^{r,s}). \tag{5.1}$$

As a consequence of (4.5, iv) and (4.6, c), it is easy to see that the operator d decomposes, under the splitting (5.1), into a sum of four operators, written ∂', $\bar{\partial}'$ ∇' and $\bar{\nabla}'$, which, in terms of the 4-fold gradation $(p, q; r, s)$, are respectively of degrees $(1, 0; 0, 0)$, $(0, 1; 0, 0)$, $(1, 0; -1, 1)$, and $(0, 1; 1, -1)$. We define

$$\begin{aligned} \mathfrak{D}' &= \partial' + \bar{\nabla}' \\ \mathfrak{D}'' &= \bar{\partial}' + \nabla'; \end{aligned} \tag{5.2}$$

the pairing of operators in (5.2) is done according to "total holomorphic" degree $p + r$. We have Laplacian operators for $\mathfrak{D}'$ and $\mathfrak{D}''$ as in (3.11). The following generalized Kähler identities hold:

(5.3) PROPOSITION (Deligne). $\square_d = 2\square_{\mathfrak{D}'} = 2\square_{\mathfrak{D}''}$.

Proof. (See [11, §2]; the generalization to complex variations of Hodge structure is direct.)

Put

$$\mathscr{B}^{P,Q}_{(2)} = \bigoplus_{\substack{p+r=P \\ q+s=Q}} \mathscr{A}^{p,q}_{(2)}(S, \mathscr{H}^{r,s}). \tag{5.4}$$

In terms of this new bigrading, $\mathfrak{D}'$ is of bidegree $(1, 0)$ and $\mathfrak{D}''$ is of bidegree $(0, 1)$. As a consequence of (5.3), one obtains a decomposition of the harmonic forms into harmonic components of type (P, Q):

$$\mathscr{h}^{n}_{(2)}(S, \mathbf{V}) = \bigoplus_{P+Q=m+n} \mathscr{h}^{P,Q}_{(2)}, \tag{5.5}$$

with $\mathscr{h}^{P,Q}_{(2)} = \overline{\mathscr{h}^{Q,P}_{(2)}}$ in the real case. This decomposition passes to cohomology, as in (3.18), and thus we have

(5.6) THEOREM (Deligne). *Let* $\mathbf{V}$ *underlie a complex polarizable variation of Hodge structure of weight* m *on* S. *Then there is, for each* n, *an associated decomposition:*

$$\bar{H}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{P+Q=m+n} \bar{H}^{P,Q}_{(2)}$$

If the variation of Hodge structure is real, then the above is a Hodge structure of weight $m+n$.

In order to work effectively with these decompositions, it is best to eliminate the C^∞ forms, and work only on the holomorphic level through the use of hypercohomology. Let $\Omega^\bullet_S(\mathbf{V})$ denote the holomorphic deRham complex with values in $\mathbf{V}$, with differential ∂. The following was given by Deligne:

(5.7) *Definition.* The *Hodge filtration* $\{F^r\,\Omega^\bullet_S(\mathbf{V})\}$ on $\Omega^\bullet_S(\mathbf{V})$ is given by

$$F^r\,\Omega^n_S(\mathbf{V}) = \Omega^n_S \otimes \mathscr{F}^{r-n};$$

because of (4.5, iv), $F^r\,\Omega^\bullet_S(\mathbf{V})$ is a sub-complex of $\Omega^\bullet_S(\mathbf{V})$.

The successive quotients

$$Gr^r_F\,\Omega^\bullet_S(\mathbf{V}) = F^r\,\Omega^\bullet_S(\mathbf{V})/F^{r+1}\,\Omega^\bullet_S(\mathbf{V})$$

have terms

(5.8) $$Gr^r_F\,\Omega^n_S(\mathbf{V}) = \Omega^n_S \otimes \mathscr{G}r^{r-n},$$

where $\mathscr{G}r^q = \mathscr{F}^q/\mathscr{F}^{q+1}$.

Then $\mathscr{A}^\bullet(S,\mathbf{V})$ is a fine resolution of $\Omega^\bullet_S(\mathbf{V})$, possessing a corresponding filtration. We summarize the main consequences:

(5.9) PROPOSITION. *Assume that* S *is compact. Then*

i) *The spectral sequence*

$$E_1^{p,q} = \mathbf{H}^{p+q}(S, Gr^p_F\,\Omega^\bullet_S(\mathbf{V})) \Rightarrow \mathbf{H}^{p+q}(S, \Omega^\bullet_S(\mathbf{V})) \simeq H^{p+q}(S, \mathbf{V})$$

degenerates at E_1.

ii) *The filtration induced by* $\{F^r\,\Omega^\bullet_S(\mathbf{V})\}$ *on* $H^n(S, \mathbf{V})$ *coincides with the Hodge filtration associated to the decomposition of (5.6) under Definition (4.4).*

iii) *There is a natural identification*

$$H^{P,Q} \simeq \mathbf{H}^n(S, Gr^P_F\,\Omega^\bullet_S(\mathbf{V})$$

for $P+Q = m+n$.

Proof. The above statements are all immediate consequences of (5.6).

We now specialize to the case of a locally homogeneous variation of Hodge structure associated to (ρ, V), as described in the preceding section. First, we recall the differential operators from (3.9) and (3.10), and observe:

(5.10) PROPOSITION. *For a locally homogeneous variation of Hodge structure,*

$$\partial' = D' \quad \textit{and} \quad \nabla' = d'_\rho .$$

(5.11) COROLLARY. *Under the same hypothesis,*

$$\mathfrak{D}' = D' + d''_\rho \quad \textit{and} \quad \mathfrak{D}'' = D'' + d'_\rho .$$

This, when coupled with (5.3), explains (3.19).

We seem to have two different Hodge decompositions on $\bar{H}^n_{(2)}(S, \mathbf{V})$, one (5.6) coming from the variation of Hodge structure, and one given by (3.18). The two, in fact, are mutually compatible, for (3.14) implies that the Laplacian $\Box_d$ respects the complete decomposition (5.1). Thus, we obtain in the locally homogeneous case

(5.12) $$\mathscr{h}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{\substack{p+q=n \\ r+s=m}} \mathscr{h}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}),$$

and thus

(5.13) $$\bar{H}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{\substack{p+q=n \\ r+s=m}} \bar{H}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}),$$

with

(5.14) $$\bar{H}^{p,q}_{(2)}(S, \mathbf{V}) = \bigoplus_{r+s=m} \bar{H}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}),$$

(5.15) $$\bar{H}^{P,Q}_{(2)} = \bigoplus_{\substack{p+r=P \\ q+s=Q}} \bar{H}^{(p,q);(r,s)}_{(2)}(S, \mathbf{V}).$$

As a consequence of (5.12), we derive

(5.16) PROPOSITION. *Assume S is compact. Then for all integers k, the spectral sequence*

$$_IE_1^{p,q} = H^q(S, \Omega^p_S \otimes \mathscr{G}_{\mathbb{Z}}{}^{k-p}) \Rightarrow \mathbf{H}^{p+q}(S, Gr^k_F \Omega^\bullet_S(\mathbf{V}))$$

degenerates at E_2.

Proof. We have the isomorphism

$$\mathbf{H}^n(S, Gr_F^k \Omega_S^\bullet(\mathbf{V})) \simeq H^{k,\, n+m-k} = \bigoplus_{\substack{p+r=k \\ q+s=n+m-k}} H^{(p,q);(r,s)}(S, \mathbf{V}),$$

and by the Hodge-Dolbeault isomorphism and [11, (1.10)]

$$\begin{aligned} {}_I E_1^{p,q} &= H^q(S, \Omega_S^p \otimes \mathcal{G}_{\mathfrak{k}}^{k-p}) \\ &\simeq \{\eta \in \mathcal{A}^{p,q}(\mathcal{H}^{k-p,\, m-k+p}) : \square_{\bar\partial'} \eta = 0\}. \end{aligned}$$

The E_2 term of the spectral sequence is equal to the cohomology of the E_1 term under its differential d_1. We again use [11, (1.10)] to assert that in terms of the $\bar\partial'$-harmonic forms, d_1 is given by ∇'. In other words, the E_2 terms are naturally isomorphic to the cohomology groups of the complex of D''-harmonic forms under the differential d'_ρ. Representing classes by d'_ρ-harmonic forms, we have

$$\oplus\, {}_I E_2^{p,q} \simeq \{\eta : \square_{D''}\eta = 0, \quad \text{and also} \quad \square_{d'_\rho}\eta = 0\}.$$

By (3.20), the right-hand side gives $H^{k,\, n+m-k}$, so the desired conclusion follows.

(5.17) COROLLARY 1. *There is a natural injection*

$$\mathcal{h}^{(p,q);(r,s)} \to H^q(S, \Omega_S^p \otimes \mathcal{G}_{\mathfrak{k}}^r)$$

(5.18) COROLLARY 2 [7, p. 413]. $H^{0,n}(S, \mathbf{V}) = H^{(0,n);(0,m)}(S, \mathbf{V})$. *(Hence also* $H^{n,0}(S, \mathbf{V}) = H^{(n,0);(m,0)}(S, \mathbf{V})$.*)*

Proof. Let η be a harmonic $(0, n)$-form with values in $\mathbf{V}$. Then by (3.20), we must have

$$d'_\rho \eta = 0.$$

Since η is an anti-holomorphic form, we than have

$$\rho(X)\eta = 0 \quad \text{for all} \quad X \in \mathfrak{p}^+.$$

This forces the form $\tilde\eta$ (3.3) to take its values in $V\langle m\rangle$, proving (5.18) by the last assertion in §4.

We also have:

(5.19) PROPOSITION. *The spectral sequence*

$${}_{II}E_2^{p,q} = H^p(S, \mathcal{H}^q(Gr_F^k \Omega_S^\bullet(\mathbf{V})) \Rightarrow \mathbf{H}^{p+q}(S, Gr_F^k \Omega_S^\bullet(\mathbf{V}))$$

degenerates at E_2 *if* S *is compact, and*

$$_{II}E_2^{p,q} \simeq H^{(q,p);(k-q,m+k-q)}(S, \mathbf{V}).$$

Proof. Let

$$\begin{aligned}
\mathscr{H}_k^q &= \mathscr{H}^q(Gr_F^k \Omega_S^\bullet(\mathbf{V})), \\
\mathscr{C}_k^q &= \Omega_S^q \otimes \mathscr{G}_{\mathfrak{k}}{}^{k-q}, \\
\mathscr{Z}_k^q &= \ker\{\Omega_S^q \otimes \mathscr{G}_{\mathfrak{k}}{}^{k-q} \to \Omega_S^{q+1} \otimes \mathscr{G}_{\mathfrak{k}}{}^{k-q-1}\} \\
\mathscr{B}_k^q &= \operatorname{im}\{\Omega_S^{q-1} \otimes \mathscr{G}_{\mathfrak{k}}{}^{k-q+1} \to \Omega_S^q \otimes \mathscr{G}_{\mathfrak{k}}{}^{k-q}\}.
\end{aligned}$$

Then all of the above are automorphic vector bundles associated to representations of K on vector spaces H_k^q, C_k^q, Z_k^q, B_k^q respectively. As a consequence of Schur's Lemma and the semi-simplicity of K-representations,

$$C_k^q \simeq H_k^q \oplus B_k^q \oplus B_k^{q+1}$$

as a representation of K. Therefore, by (2.7),

$$\mathscr{C}_k^q \simeq \mathscr{H}_k^q \oplus \mathscr{B}_k^q \oplus \mathscr{B}_k^{q+1}.$$

This implies that there is an embedding

$$\oplus_q \mathscr{H}_k^q \to Gr_F^k \Omega_S^\bullet(\mathbf{V}),$$

which is a quasi-isomorphism. From this, it is clear that the spectral sequence $_{II}E_2^{p,q}$ must degenerate at E_2, and moreover that $H^p(S, \mathscr{H}_k^q)$ gives the (q, p); $(k-q, m+q-k)$ component of $H^{p+q}(S, \mathbf{V})$.

(5.20) *Remark.* We also obtain from the above that

$$_{I}E_2^{p,q} \simeq {}_{II}E_2^{q,p},$$

so the argument of (5.19) gives an alternate proof of (5.16).

By combining (5.19) with (5.9), we obtain

(5.21) Corollary. *If* S *is compact,*

$$\dim H^n(S, \mathbf{V}) = \sum_k \sum_{p+q=n} \dim H^p(S, \mathscr{H}_k^q).$$

We can generalize (5.19) to the non-compact case, if we forego the hypercohomology. Because a morphism of representations of K induces a *bounded* mapping between the associated locally homogeneous vector bundles, we can see, by reasoning similar to that used in (5.19), that

$$\text{(5.22)} \qquad H_{(2)}^{q,\,p;\,k-q,\,m+q-k}(S,\mathbf{V}) \simeq {}_{\bar\partial}H_{(2)}^{p}(S,\mathscr{H}_k^q)$$

$$=: \frac{\{\phi \in \mathscr{A}_{(2)}^{0,\,p}(\mathscr{H}_k^q) : \bar\partial\phi = 0\}}{\{\phi \text{ as above: } \phi = \bar\partial\eta \text{ for some } \eta \in \mathscr{A}_{(2)}^{0,\,p-1}(\mathscr{H}_k^q)\}}$$

If we make use of the full extent of (3.29), we can actually deduce the following generalization of (5.9):

(5.23) THEOREM. *Let $Y^\bullet$ be a Q-invariant $d'_\mathfrak{p}$-subcomplex of $\Lambda^\bullet\mathfrak{p}^- \otimes V$ (Q as in (1.10)), such that $\mathfrak{p}^- \wedge Y^p \subset Y^{p+1}$, so that a sub-complex $\mathscr{Y}^\bullet$ of holomorphic sub-bundles of $\Omega_S^\bullet(\mathbf{V})$ is determined by $Y^\bullet$. Assume also that*

$$Y^\bullet \cap d'_\mathfrak{p}(\Lambda^\bullet\mathfrak{p}^- \otimes V) = d'_\mathfrak{p}\,Y^\bullet.$$

Then if S is compact, there are short exact sequences

$$0 \to \mathbf{H}^n(S,\mathscr{Y}^\bullet) \xrightarrow{\iota} H^n(S,\mathbf{V}) \to \mathbf{H}^n(S,\Omega_S^\bullet(\mathbf{V})/\mathscr{Y}^\bullet) \to 0$$

for all n, with $\iota(\mathbf{H}^n(S,\mathscr{Y}^\bullet))$ given by the subspace of harmonic n-forms with values in $\mathscr{Y}^\bullet$.

Proof. Let $F^p\mathscr{Y}^\bullet$ be the filtration on $\mathscr{Y}$ induced by (5.7). Consider the spectral sequences

$${}_AE_1^{p,\,q} = \mathbf{H}^{p+q}(S, Gr_F^p\mathscr{Y}^\bullet) \Rightarrow \mathbf{H}^{p+q}(S,\mathscr{Y}^\bullet),$$

$${}_BE_1^{r,\,s} = \mathbf{H}^s(S, Gr_F^p\mathscr{Y}^r) \Rightarrow \mathbf{H}^{r+s}(S, Gr_F^p\mathscr{Y}^\bullet).$$

As in (5.16), the second one degenerates at E_2, with

$${}_BE_2^{r,\,s} \simeq H^s(S, \mathscr{H}^r Gr_F^p\mathscr{Y}^\bullet.$$

By assumption, $\mathscr{H}^p\, Gr_F^p\mathscr{Y}^\bullet$ may be identified with an equivariant *sub-bundle* of $\mathscr{H}^r_p$, whence ${}_BE_2^{r,\,s}$ becomes identified with a subspace of $\mathscr{h}^{(r,\,s);\,(p-r,\,m-p+r)}(S,\mathbf{V})$. Thus, the mapping

$${}_AE_1^{p,\,q} \to \mathbf{H}^{p+q}(S, Gr_F^p\Omega_S^\bullet\ (\mathbf{V}))$$

is an injection. Since the spectral sequence (5.9, i) degenerates at E_1, it follows that ${}_AE^{p,\,q}$ does likewise, and the assertions of (5.23) follow.

(5.24) *Remark.* Take $\mathscr{Y}^\bullet = (F^p\Omega_S^\bullet) \otimes \mathbf{V}$ (any p) in (5.23). Then one recovers (3.18) and its algebraic consequences: the spectral sequence

$$E_1^{p,\,q} = H^q(S,\Omega_S^p(\mathbf{V})) \Rightarrow \mathbf{H}^{p+q}(S,\Omega_S^\bullet(\mathbf{V})) \simeq H^{p+q}(S,\mathbf{V})$$

degenerates at E_1.

We next analyze the terms of the Hodge decomposition, as given in (5.9, iii). In particular, we will concern ourselves with the vanishing of some of these terms.

Given the irreducible representation (ρ, V) of G, we let τ_p denote the representation of K on the subspace $H_0^{p,q}$ (4.9) of V obtained by restricting ρ. The following is an immediate consequence of our constructions:

(5.25) LEMMA. *There is a holomorphic isomorphism*

$$\mathcal{G}_{\mathcal{E}}{}^p \simeq E(\Gamma, \tau_p).$$

(5.26) COROLLARY. *As holomorphic vector bundles on S, the terms of the complex $Gr_F^k \Omega_S^\bullet(\mathbf{V})$ are*

$$\Omega_S^p \otimes \mathcal{G}_{\mathcal{E}}{}^{k-p} \simeq E(\Gamma, \Lambda^p \mathrm{Ad}^- \otimes \tau_{k-p}).$$

(5.27) COROLLARY. *Assume S is compact. Then for all n, $H^{n,0}(S, \mathbf{V})$ is given by the space of automorphic forms*

$$\{f \in \Gamma(M, \Omega_M^0) \otimes \Lambda^n \mathfrak{p}^- \otimes H_0^{m,0} : f(\gamma x) = (\Lambda^n \mathrm{Ad}^- \otimes \tau_m)\big(\mathcal{J}(\gamma, x)\big) \cdot f(x)\}$$

Proof. Combine (5.18) and (5.26) with (2.13).

Establishing the vanishing of some of the $H^{P,Q}$ is easiest if we can prove that the complex $Gr_F^P \Omega_S^\bullet(\mathbf{V})$ is acyclic, or is at least close to being so (cf. [11, §12]). Since the differentials in this complex are $\mathcal{O}_S$-linear, we are reduced to a problem of linear algebra. We make the following simple observation:

(5.28) LEMMA. *Under the identification (5.26), the differentials in $Gr_F^p \Omega_S^\bullet(\mathbf{V})$ are given by $d_\mathfrak{p}'$ (3.10).*

As was pointed out to me by David DeGeorge, the operator $d_\mathfrak{p}'$, when applied to *all* of $\Lambda^\bullet \mathfrak{p}^- \otimes V$, gives rise to the Lie algebra cohomology $H^\bullet(\mathfrak{p}^+, V)$ for the Abelian Lie algebra $\mathfrak{p}^+$. We have the C^∞ isomorphism

$$E(\Gamma, \Lambda^\bullet \mathrm{Ad}^- \otimes \rho) \simeq \bigoplus_k Gr_F^k \Omega_S^\bullet(\mathbf{V}).$$

Moreover, we can recover each summand $Gr_F^k \Omega_S^\bullet(\mathbf{V})$, since the central subgroup Δ (defined after (1.8)) of K acts on $\Lambda^p \mathfrak{p}^- \otimes H^{k-p, m-k+p}$ by the character

$$\chi_{-p\mu + [\lambda - (k-p)\mu]} = \chi_{\lambda - k\mu}$$

independent of p, and faithfully determined by k.

Since $d_{\mathfrak{p}}'$ commutes with the action of K (see, e.g., [10, (2.5.1.1)]), $H^q(\mathfrak{p}^+, V)$ is a representation space for K, and equals the direct sum of certain irreducibles contained in the q-cochains (cf. the proof of (5.19)). Let σ_l^q denote the representation of K on $H^q(\mathfrak{p}^+, V) < l >$. Since the construction

$$\tau \mapsto E(\Gamma, \tau)$$

is an exact functor, we combine the above statements to obtain

(5.29) THEOREM. *The cohomology sheaves of* $Gr_F^k \Omega_S^{\bullet}(V)$ *are given by*

$$\mathscr{H}_k^q \simeq E(\Gamma, \sigma_{\lambda-k\mu}^q).$$

(5.30) *Remark.* This explains the occurrence of Lie algebra cohomology in [8, §10]. Also, we point out that (5.18) can be deduced from (5.29).

In order to compute the cohomology sheaves in (5.29), we will make use of a result due to Kostant [6]. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}_{\mathbf{C}}$, which we may take to be contained in $\mathfrak{k}_{\mathbf{C}}$. Let Ψ denote the set of roots for $\mathfrak{g}_{\mathbf{C}}$ relative to $\mathfrak{h}$. Let Ψ_1 denote the set of compact roots, i.e., the roots α for which the associated eigenspace $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\mathbf{C}}$ is contained in $\mathfrak{k}_{\mathbf{C}}$; let Ψ_2 denote the set of complementary (non-compact) roots. It is possible to choose the positive Weyl chamber in $\mathfrak{h}^*$ so that for the set of positive complementary roots Ψ_2^+,

$$\mathfrak{p}^+ = \bigoplus_{\alpha \in \Psi_2^+} \mathfrak{g}_\alpha .$$

Then

$$\mathfrak{q} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+$$

is a parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$, with $\mathfrak{p}^+$ its nilpotent radical, and $\mathfrak{k}_{\mathbf{C}}$ a Levi subalgebra of $\mathfrak{q}$. We also define Ψ^+, Ψ^-, and Ψ_1^+ in the obvious way. Then for our set-up, the theorem of [6] states

(5.31) THEOREM (Kostant). *For a dominant integral weight* $\Lambda \in \mathfrak{h}^*$, *let* V_Λ *be the irreducible representation of* G *with highest weight* Λ. *Then as a representation of* K,

$$H^q(\mathfrak{p}^+, V_\Lambda) \simeq \bigoplus_{w \in W_u(q)} E_{w(\Lambda+\delta)-\delta}$$

where $\delta = \frac{1}{2}\sum_{\alpha \in \Psi^+} \alpha$, E_β *is the representation of* K *with highest weight* β, *and* $W_u(q)$ *denotes the subset of the Weyl group for* $\mathfrak{h}$ *consisting of those* *elements* w *which move exactly* q *elements of* Ψ^- *into* Ψ_2^+, *but no elements of* Ψ^- *into* Ψ_1^+. [1])

(5.32) COROLLARY: $\mathscr{H}_k^q$ *is a holomorphic vector bundle with fiber isomorphic to*

$$\bigoplus_{w \in W_u(q)} E_{w(\Lambda+\delta)-\delta} <\lambda - k\mu> .$$

Since E_β is an irreducible representation of K, the subgroup Δ of K acts with a single character $\chi_{n(\beta)}$, which may be determined from the highest weight itself. In fact, it is clear that $\beta \mapsto n(\beta)$ extends to a linear functional on $\mathfrak{h}^*$, given by evaluation on a uniquely determined element of $\mathfrak{z}$. (This point will also be used in the discussion of vanishing theorems at the end of this section.) Therefore, we may rewrite the terms in the formula of (5.32) as

$$E_{w(\Lambda+\delta)-\delta} <\lambda - k\mu> = \begin{cases} E_{w(\Lambda+\delta)-\delta} & \text{if } \lambda - k\mu = n[w(\Lambda+\delta)-\delta] \\ 0 & \text{otherwise} . \end{cases}$$

(5.33) *Example.* In the case $G = SL(2, \mathbf{R})$, $\Psi_1 = \emptyset$, $\mathfrak{h}$ is one-dimensional, and the highest weights can be identified with the non-negative integers. Let $V_m = \mathrm{Symm}^m(\mathbf{C}^2)$, and let ρ_m be the corresponding representation of G. Then $\rho_2 \simeq \mathrm{Ad}$, so under the identification with integers, $\delta = 1$. Moreover, the Weyl group decomposes as the identity element I in $W_u(0)$, and $-I$ in $W_u(1)$. Thus, we obtain

$$\mathscr{H}^0(Gr_F^k\, \Omega_S^\bullet(\mathbf{V}_m)) = E_m < m - 2k >$$
$$\mathscr{H}^1(Gr_F^k\, \Omega_S^\bullet(\mathbf{V}_m)) \simeq E_{-m-2} < m - 2k > ,$$

as $\mu = 2$. But for $SL(2, \mathbf{R})$, $Z = K$, and therefore

$$E_m < n > = \begin{cases} E_m & \text{if } n = m \\ 0 & \text{if } n \neq m . \end{cases}$$

Inserting this above, we see that $Gr_F^k\, \Omega_S^\bullet(\mathbf{V}_m)$ is acyclic except for $k = 0$ (where $\mathscr{H}^0 \neq 0$) and $k = m + 1$ (where $\mathscr{H}^1 \neq 0$); this yields the Shimura isomorphism, cf. [11, (12.14)].

The above gives rise to an interesting approach to cohomology vanishing theorems for real representations, like those of [8, Thms. (8.2), (12.1)]—there, however, no realness hypothesis is imposed on the representation. The idea is

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[1]) $\cup_q W(q)$ is a set of representatives for the Weyl group of $\mathfrak{g}_\mathbf{C}$ modulo that of $\mathfrak{k}_\mathbf{C}$, consisting of those elements which keep the positive chamber for $\mathfrak{g}_\mathbf{C}$ inside the larger positive chamber for $\mathfrak{k}_\mathbf{C}$.

relatively simple: if cohomology occurs in multi-degree $(p, q; k, m-k)$, then (by conjugation) it must also occur in multi-degree $(q, p; m-k, k)$.

(5.34) PROPOSITION. *Let* Λ *be the highest weight of the real representation* (ρ, V) *of the group* G. *Then a necessary condition that* $H^{p,q}(S, \mathbf{V}) \neq 0$ *is that there exist* $w_1 \in W_u(p)$ *and* $w_2 \in W_u(q)$ *such that* $n[w_1\Lambda + w_2\Lambda] = 0$.

Proof. By (5.32),

$$H^{(p,q);(k-p,m+p-k)}(S, \mathbf{V}) \simeq H^q(S, \mathscr{H}_k^p) = 0$$

unless there exists $w_1 \in W_u(p)$ with

$$n[w_1(\Lambda+\delta)-\delta] = \lambda - k\mu. \tag{5.35}$$

(Note that $\lambda = n(\Lambda)$ and $\mu = n(\Xi)$, where Ξ denotes the highest weight of Ad.)

Similarly, for the conjugate term

$$H^{(q,p);(m+p-k,k-p)}(S, \mathbf{V}) \simeq H^p(S, \mathscr{H}_{m+p+q-k}^q)$$

to be non-zero, we must also have for some $w_2 \in W_u(q)$

$$n[w_2(\Lambda+\delta)-\delta] = \lambda - (m+p+q-k)\mu. \tag{5.36}$$

Adding (5.35) and (5.36), we see that

$$\dim H^q(S, \mathscr{H}_k^p) = \dim H^p(S, \mathscr{H}_{m+p+q-k}^q) = 0$$

unless there exist $w_1 \in W_u(p)$ and $w_2 \in W_u(q)$ such that

$$n[w_1(\Lambda+\delta)-\delta] + n[w_2(\Lambda+\delta)-\delta] = 2\lambda - (m+p+q)\mu. \tag{5.37}$$

We use the identities

$$n[\delta - w\delta] = q\mu \quad \text{if} \quad w \in W_u(q),$$
$$2\lambda = m\mu$$

to rewrite (5.37) as

$$n[w_1\Lambda] + n[w_2\Lambda] = 0, \tag{5.38}$$

or

$$n[w_1\Lambda + w_2\Lambda] = 0.$$

Let $\zeta = \sum_{\alpha \in \Psi_2^+} \alpha$. It is easily checked that under the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$ via the Killing form, ζ represents a non-zero element of $\mathfrak{z}$. Thus, the condition (5.38) can be rewritten as

$$<w_1\Lambda, \zeta> + <w_2\Lambda, \zeta> = 0,$$

or

$$<\Lambda, w_1^{-1}\zeta + w_2^{-1}\zeta> = 0. \tag{5.39}$$

We are discussing real representations, for which $w_0\Lambda = -\Lambda$ (by self-contragredience) with w_0 denoting the (unique) element of the Weyl group which maps Ψ^+ to Ψ^-. It suffices then to examine the sums

$$w_1^{-1}\zeta - w_0 w_2^{-1}\zeta \qquad w_1 \in W_u(p), w_2 \in W_u(q).$$

For instance, the assertion that $H^{p,q}(\Gamma; \rho, V) = 0$ if $p + q \neq \dim S$ and Λ lies in the interior of the dominant cone [8, Thm. (12.1)] would follow from the corresponding statement: $w_1^{-1}\zeta - w_0 w_2^{-1}\zeta$ is a non-zero element of the positive dual cone for all $w_1 \in W_u(p)$ and $w_2 \in W_u(q)$ whenever $p + q < \dim S$. This has been verified by the author in some examples, though no satisfying argument involving root structure has been found.

Added in proof: Borel has pointed out that although there was a gap in the proof of Theorem 4 of [13], it has been filled in by W. Casselman in the case where the ranks of G and K are equal. This includes all Hermitian cases.

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