III. A VARIANT

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Let us agree to call a scheme S *accessible* if there exists an absolutely finitely generated field K for which the set $S(K)$ of K-valued points of S is non-empty. Thus for example, if K is an absolutely finitely generated field, then for *any* subring $R \subset K$, Spec (R) is accessible (by the K-valued point $R \hookrightarrow K$); also any subring R' of the power-series ring $K[[X_1, ..., ...]]$ over K in any number of variables has Spec (R') accessible

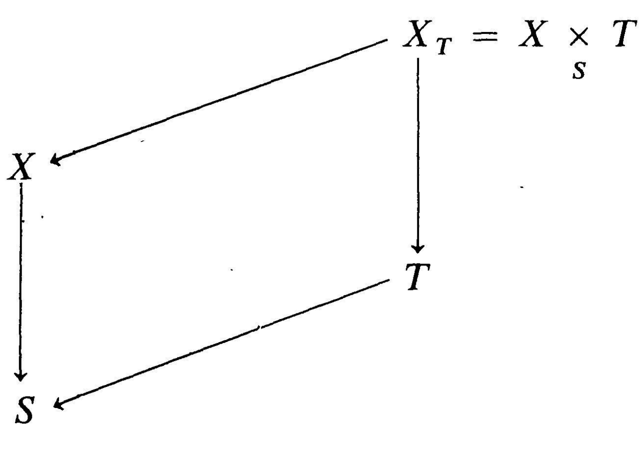
$$(\text{by } R' \hookrightarrow K[[X_1, ...]] \xrightarrow{X \to 0} K).$$

On the other hand, the spectrum of a field F is accessible if and only if F is absolutely finitely generated.

THEOREM 2. *Let S be a connected, locally noetherian scheme which is accessible. Let X/S be a proper and smooth S-scheme with geometrically connected fibres. Then the group* Ker (X/S) *is finite.*

Proof. We begin by reducing to the case when S is a finitely generated field. In view of the accessibility of S, this reduction results from the following simple lemma applied with T = Spec (K).

LEMMA 4. *Let X/S be proper and smooth with geometrically connected fibres over a connected locally noetherian scheme S. Given a connected locally noetherian S-scheme T, denote by X_T/T the inverse image of X/S on T, i.e. form the cartesian diagram*



The natural map (cf. 1.5)

$$\mathrm{Ker}\,(X_T/T) \to \mathrm{Ker}\,(X/S)$$

is surjective.

Proof. Let t be a geometric point of T, s the image geometric point of S, and x a geometric point on the fibre X_s. The homotopy exact sequences (SGA I, Exp X, 1.4) for X/S and X_T/T sit in a commutative diagram

$$\begin{array}{ccccccc}
\pi_1(X_s, x) & \longrightarrow & \pi_1(X_T, x) & \longrightarrow & \pi_1(T, t) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\pi_1(X_s, x) & \longrightarrow & \pi_1(X, x) & \longrightarrow & \pi_1(S, s) & \longrightarrow & 0 .
\end{array}$$

Passing to the abelianizations yields the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\pi_1(X_s)^{ab} & \longrightarrow & \pi_1(X_T)^{ab} & \longrightarrow & \pi_1(T)^{ab} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\pi_1(X_s)^{ab} & \longrightarrow & \pi_1(X)^{ab} & \longrightarrow & \pi_1(S)^{ab} & \longrightarrow & 0
\end{array}$$

whence we find

$$\begin{array}{ccl}
 & \nearrow\!\!\!\!\!\rightarrow & \mathrm{Ker}\,(X_T/T) = \text{image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X_T)^{ab} . \\
\pi_1(X_s)^{ab} & & \quad\downarrow \\
 & \searrow\!\!\!\!\!\rightarrow & \mathrm{Ker}\,(X/S) \;\; = \text{image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X)^{ab} .
\end{array}$$

QED

Thus we are reduced to proving the finiteness of Ker (X/K) when K is an absolutely finitely generated field, and X/K is proper, smooth, and geometrically connected. We have already proven this finiteness theorem when X/K is an abelian variety (cf. Remark (1) above). We will reduce to this case by making use of the theory of the Picard and Albanese varieties.

At the expense of replacing K by a finite extension, we may assume that X has a K-rational point x_0. The Picard scheme $Pic_{X/K}$ is then a commutative group-scheme locally of finite type over K, which represents the functor on $\{\text{Schemes}/K\}$

$$W \to \left\{ \begin{array}{l} \text{the group of } W\text{-isomorphism classes of pairs } (\mathscr{L}, \varepsilon) \text{ consisting} \\ \text{of an invertible sheaf } \mathscr{L} \text{ on } X \underset{K}{\times} W \text{ together with a} \\ \text{trivialization } \varepsilon \text{ of the restriction } \mathscr{L} \text{ to } \{x_0\} \underset{K}{\times} W \end{array} \right.$$

The subgroup-scheme $Pic^{\tau}_{X/K}$ of $Pic_{X/K}$ classifies those $(\mathscr{L}, \varepsilon)$ whose underlying $\mathscr{L}$ becomes τ-equivariant to zero when restricted to every geometric fibre of $X \times W/W$ (i.e. for each geometric point w of W, some multiple of $\mathscr{L} \mid X \times w$ is algebraically equivalent to zero). The identity component $Pic^{0}_{X/K}$ of $Pic_{X/K}$ classifies those $(\mathscr{L}, \varepsilon)$ whose $\mathscr{L}$ becomes algebraically equivalent to zero on each geometric fibre $X \times W/W$. The Picard *variety* $Pic^{0,\,\mathrm{red}}_{X/K}$ is an abelian variety over K, and it sits in an *f.p.p.f.* short exact sequence of commutative group schemes

$$(3.1) \qquad 0 \to Pic^{0,\,\mathrm{red}}_{X/K} \to Pic^{\tau}_{X/K} \to C \to 0$$

in which the cokernel C is a finite flat group-scheme over K. This cokernel C should be thought of as the "scheme theoretic" torsion in the Neron-Severi group.

We denote by $\mathrm{Alb}_{X/K}$ the Albanese variety of X/K, *defined* to be the dual abelian variety to the Picard variety $Pic^{0,\,\mathrm{red}}_{X/K}$. We now recall the expression of $\pi_1 (X \otimes \overline{K})^{ab}$ in terms of the Tate module of the Albanese, and a finite "error term" involving the Cartier dual $C^{\vee}$ of C.

LEMMA 5. *Let* K *be a field, and* X/K *a proper, smooth and geometrically connected* K*-scheme which admits a* K*-rational point. Then there is a canonical short exact sequence of* $\mathrm{Gal}(\overline{K}/K)$*-modules*

$$(3.2) \qquad 0 \to C^{\vee}(\overline{K}) \to \pi_1 (X \otimes \overline{K})^{ab} \to T\left(\mathrm{Alb}_{X/K}(\overline{K})\right) \to 0 .$$

Proof. By Kummer and Artin-Schreier theory, we have for each integer $N \geqslant 1$ a canonical isomorphism

$$\begin{aligned} &\mathrm{Hom}\left(\pi_1 (X \otimes \overline{K})^{ab}, \mathbf{Z}/N\mathbf{Z}\right) \\ &= H^1_{et}(X \otimes \overline{K}, \mathbf{Z}/N\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}\left(\mu_N, (Pic^{\tau}_{X/K}) \otimes \overline{K}\right). \end{aligned}$$

in which the last Hom is in the sense of $\overline{K}$-group-schemes. Applying the functor $X \mapsto \mathrm{Hom}(\mu_N, X)$ to the short exact sequence

$$0 \to Pic^{0,\,\mathrm{red}} \to Pic^{\tau} \to C \to 0$$

gives a short exact sequence

$$\begin{aligned} (3.3) \qquad & 0 \to \mathrm{Hom}\left(\mu_N, (Pic^{0,\,\mathrm{red}}) \otimes \overline{K}\right) \\ & \to \mathrm{Hom}\left(\mu_N, (Pic^{\tau}) \otimes \overline{K}\right) \to \mathrm{Hom}\left(\mu_N, C \otimes \overline{K}\right) \to 0 \end{aligned}$$

(the final zero because over an algebraically closed field, the group $\mathrm{Ext}^1(\mu_N, A)$ vanishes for any abelian variety A, cf. the remark at the end of this section). We now "decode" its two end terms, using Cartier-Nishi duality for the first, and Cartier duality for the last.

The first is

$$\begin{aligned}\mathrm{Hom}\left(\mu_N, (Pic^{0,\mathrm{red}}) \otimes \bar{K}\right) &= \mathrm{Hom}\left(\mu_N, (Pic^{0,\mathrm{red}})_N \otimes \bar{K}\right)\\ &\quad \wr\downarrow \quad \text{Cartier-Nishi duality}\\ &\quad \mathrm{Hom}\left(\mathrm{(Alb}_{X/N})_N \otimes \bar{K}, \mathbf{Z}/N\mathbf{Z}\right)\\ &\quad \wr\downarrow \quad \text{evaluation on } \bar{K}\text{-points}\\ &\quad \mathrm{Hom}\left((\mathrm{Alb}_{X/K}(\bar{K}))_N, \mathbf{Z}/N\mathbf{Z}\right)\\ &\quad \wr\downarrow\\ &\quad \mathrm{Hom}\left(T(\mathrm{Alb}_{X/K}(\bar{K})), \mathbf{Z}/N\mathbf{Z}\right).\end{aligned}$$

The last is

$$\mathrm{Hom}(\mu_N, C \otimes \bar{K}) \xrightarrow[\text{Cartier duality}]{\sim} \mathrm{Hom}\left(C^\vee \otimes \bar{K}, \mathbf{Z}/N\mathbf{Z}\right)$$
$$\downarrow\wr \quad \text{evaluation}$$
$$\mathrm{Hom}\left(C^\vee(\bar{K}), \mathbf{Z}/N\mathbf{Z}\right)$$

"Substituting" into the exact sequence (3.2), we find a canonical short exact sequence

(3.4)
$$0 \to \mathrm{Hom}\left(T(\mathrm{Alb}_{X/K}(\bar{K})), \mathbf{Z}/N\mathbf{Z}\right)$$
$$\to \mathrm{Hom}\left(\pi_1(X \otimes \bar{K})^{ab}, \mathbf{Z}/N\mathbf{Z}\right) \to \mathrm{Hom}\left(C^\vee(\bar{K}), \mathbf{Z}/N\mathbf{Z}\right) \to 0$$

Passing to the *direct* limit as N grows multiplicatively, we obtain a canonical short exact sequence

(3.5)
$$0 \to \mathrm{Hom}\left(T(\mathrm{Alb}_X(\bar{K}), \mathbf{Q}/\mathbf{Z}\right)$$
$$\to \mathrm{Hom}\left(\pi_1(X \otimes \bar{K})^{ab}, \mathbf{Q}/\mathbf{Z}\right) \to \mathrm{Hom}\left(C^\vee(\bar{K}), \mathbf{Q}/\mathbf{Z}\right) \to 0.$$

Taking its Pontryagin dual, we find the required exact sequence (3.2). QED

To complete the reduction of Theorem 2 to the case of abelian varieties, we simply notice that the exact sequence of lemma 5 yields, upon passage to coinvariants, an exact sequence

$$(3.6) \qquad (C^\vee(\bar{K}))_{\mathrm{Gal}(\bar{K}/K)} \to \mathrm{Ker}\,(X/K) \to \mathrm{Ker}\,(\mathrm{Alb}_{X/K}/K) \to 0$$

whose first term, being a quotient of the finite group $C^\vee(\bar{K})$, is finite. QED

Remark. In the course of the proof of Lemma 5, we appealed to the "well-known" vanishing of $\mathrm{Ext}^1(\mu_N, A)$ over an algebraically closed field, for an abelian variety A and any integer $N > 1$. Here is a simple proof. It is enough to prove this vanishing when N is either prime to the characteristic p of K, or, in case $p > 0$, when $N = p$.

Suppose first N prime to p. Because the ground-field is algebraically closed, we have $\mu_N \simeq \mathbf{Z}/N\mathbf{Z}$, so it is equivalent to prove the vanishing of $\mathrm{Ext}^1(\mathbf{Z}/N\mathbf{Z}, A)$. We will prove that *this* group vanishes for every integer $N > 1$. Consider such an extension:

$$0 \to A \to E \to \mathbf{Z}/N\mathbf{Z} \to 0$$

Pass to $\bar{K}$-valued points

$$0 \to A(\bar{K}) \to E(\bar{K}) \to \mathbf{Z}/N\mathbf{Z} \to 0$$

and consider the endomorphism "multiplication by N". Because the group $A(\bar{K})$ is N-divisible, the snake lemma gives an exact sequence

$$0 \to A(\bar{K})_N \to E(\bar{K})_N \to \mathbf{Z}/N\mathbf{Z} \to 0$$

But a point in $E(\bar{K})_N$ which maps onto "1" $\in \mathbf{Z}/N\mathbf{Z}$ is precisely a splitting of our extension.

Next consider the case $N = p = \mathrm{char}\,(K)$. We give a proof due to Barry Mazur. Using the *f.p.p.f.* exact sequence

$$0 \to A_p \to A \to A \to 0\,.$$

to compute $\mathrm{Ext}(\mu_p, -)$, we obtain a short exact sequence

$$0 \to \mathrm{Hom}\,(\mu_p, A) \to \mathrm{Ext}^1(\mu_p, A_p) \to \mathrm{Ext}^1(\mu_p, A) \to 0$$

To prove that $\mathrm{Ext}^1(\mu_p, A) = 0$, we will show that the groups $\mathrm{Hom}\,(\mu_p, A)$ and $\mathrm{Ext}^1(\mu_p, A_p)$ are both finite, of the same order. Trivially, we have $\mathrm{Hom}\,(\mu_p, A) = \mathrm{Hom}\,(\mu_p, A_p)$. Because we are over an algebraically closed field, and A_p is killed by p, its toroidal biconnected-etale decomposition looks like

$$A_p \simeq (\mu_p)^a \times (\text{biconnected}) \times (\mathbf{Z}/p\mathbf{Z})^b; \qquad [\text{in fact } a = b]\,.$$

Only the $\mu\, p$'s in A_p can "interact" with μ_p. Thus we are reduced to showing that $\mathrm{Hom}\,(\mu_p, (\mu_p)^a)$ and $\mathrm{Ext}^1(\mu_p, (\mu_p)^a)$ are both finite of the same cardinality p^a.

By Cartier duality, it is equivalent to show that both Hom $(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ and $\mathrm{Ext}^1(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ have order p, and this is obvious (resolve the "first" $\mathbf{Z}/p\mathbf{Z}$ by

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0).$$

For another proof in this case, cf. Oort, [10], 85.

IV. Absolute Finiteness theorems

Theorem 3. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbf{Q}$. Let X be a smooth $\mathcal{O}$-scheme of finite type whose geometric generic fibre $X \underset{\mathcal{O}}{\otimes} \overline{K}$ is connected, and which maps surjectively to* Spec $(\mathcal{O})$ *(i.e. for every prime $\mathfrak{p}$ of $\mathcal{O}$, the fibre over $\mathfrak{p}$, $X \underset{\mathcal{O}}{\otimes} (\mathcal{O}/\mathfrak{p})$, is non empty). Then the group $\pi_1(X)^{ab}$ is finite.*

Proof. This follows immediately from Theorem 1 and global classfield theory, according to which $\pi_1(\mathrm{Spec}\,(\mathcal{O}))^{ab}$, the galois group of the maximal unramified abelian extension of K, is *finite.* QED

Theorem 4. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbf{Q}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ a finite set of primes of $\mathcal{O}$, $N = p_1 \ldots p_n$ the product of their residue characteristics, and $\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]$ the ring of "integers outside $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$" in K. Let X be a smooth $\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]$-scheme of finite type, whose geometric generic fibre $X \underset{\mathcal{O}}{\otimes} \overline{K}$ is connected, and which maps surjectively to* Spec $(\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n])$ *(i.e. for every prime $\mathfrak{p} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, the fibre*

$$X \otimes (\mathcal{O}/\mathfrak{p})$$

is non-empty). Then the group $\pi_1(X)^{ab}$ is the product of a finite group and a pro-N group.

Proof. Again an immediate consequence of Theorem 1 and global classfield theory, according to which $\pi_1(\mathrm{Spec}\,(\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]))^{ab}$, the galois group of the maximal abelian, unramified outside $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$-extension of K is finite times pro-N. QED