

# 3. The angular defect in higher dimensions

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Theorem 1 exhibits a remarkable fact about the total angular defect of  $P$ . For, quite apart from the precise relationship between  $\Delta$  and  $\chi$  which it expresses, it shows that  $\Delta(P)$  depends only on the topological type of  $P$ . It would be remarkable enough that  $\Delta(P)$  is independent of the cellular subdivision of  $P$ ; but, in fact, it does not vary if  $P$  is replaced by some other polyhedron homeomorphic to  $P$ . Thus  $\Delta(P)$  may be said, paradoxically, to be defined by the geometry of  $P$ —and to be independent of that geometry! In fact the situation is even more remarkable, since the Euler characteristic is not only a topological invariant but even a *homotopy* invariant; this means that we may deform  $P$  continuously without changing  $\chi(P)$ —and thus without changing  $\Delta(P)$ .

### 3. THE ANGULAR DEFECT IN HIGHER DIMENSIONS

We look now at the possibility of obtaining a formula for the total angular defect for a polyhedron of arbitrary dimension. We will largely confine attention to *polytopes* (see [3]), that is, homeomorphs of <sup>1)</sup>  $S^{n-1}$ , for some  $n \geq 3$ . As explained in the Introduction, we will no longer expect to find any significant relationship with the Euler characteristic, so we will concentrate on the question of whether, for such a polytope  $P$ , we may obtain a formula for  $\Delta(P)$  in terms of  $V$ ,  $E$  and  $F$ . Our first result is very general, but will prove to be applicable for certain standard polytopes.

**THEOREM 2.** *Let  $P$  be an arbitrary polyhedron in which every edge is incident with precisely  $q$  faces, then*

$$\Delta(P) = \pi(2V - qE + 2F). \quad (3.1)$$

*Proof.* We have only to make a small modification of Pólya's argument. We proceed as in the proof of Theorem 1 as far as the relation (2.2). But now

$$\sum_{j=1}^F S_j = qE,$$

so that (2.2) implies that

$$q\pi E - 2\pi F = 2\pi V - \Delta,$$

from which (3.1) immediately follows.

<sup>1)</sup> We explain later in the section why it is more convenient to talk of  $S^{n-1}$  than of  $S^n$ .

The restriction in the hypothesis of Theorem 2, that every edge be incident with precisely  $q$  faces, is very severe, except in the case that  $P$  is 2-dimensional. What is remarkable is that it is satisfied in the case of three standard polytopes. These we now describe. In doing so it will be convenient sometimes to adopt the notation of the Introduction, replacing  $V, E, F$  by  $N_0, N_1, N_2$ , and, generally, using  $N_i$  to designate the number of cells of dimension  $i$  in the polytope  $P$ .

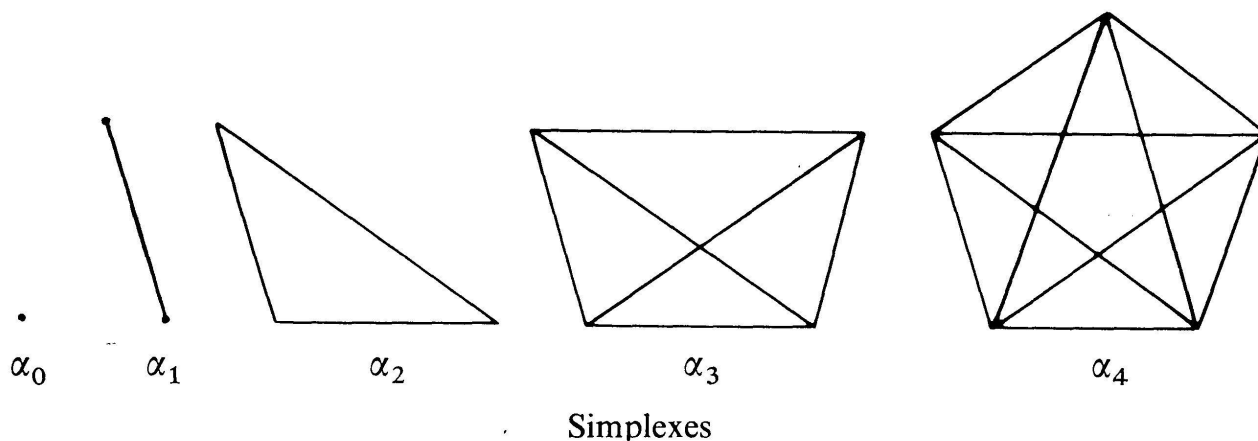


FIGURE 5

SIMPLEXES are produced, as illustrated in Figure 5, by beginning with a single point  $\alpha_0$ ; we then take this existing structure, introduce another point and join it to the previous one, thus producing  $\alpha_1$  (an edge or line segment); again, we begin with this existing structure, introduce a single point, not in the linear space spanned by  $\alpha_1$ , and join it to each of the existing points to produce  $\alpha_2$  (a triangle or 2-simplex); we continue by taking the structure of  $\alpha_2$ , introducing a single point, not in the linear space spanned by  $\alpha_2$ , and joining that point to each of the existing points to obtain  $\alpha_3$  (a tetrahedron or 3-simplex); etc. In the general case the  $(n+1)$  points we have introduced are the vertices of an  $n$ -dimensional *simplex*, or  $n$ -simplex,  $\alpha_n$ , whose cells are themselves simplexes formed by subsets of the  $(n+1)$  points, so that there are  $\binom{n+1}{1}$  vertices,  $\binom{n+1}{2}$  edges,  $\binom{n+1}{3}$  faces,  $\binom{n+1}{4}$  tetrahedra, etc. Hence we see that, if  $P_1$  is the boundary of  $\alpha_n$ ,

$$N(P_1)_k = \binom{n+1}{k+1}, \quad 0 \leq k \leq n-1.$$

When all the edges are equal these structures are called *regular simplexes*, in [3] denoted  $\alpha_i$ . The  $\alpha_i$  of Figure 5 should be viewed as though they are in perspective since they were intentionally drawn to show a symmetric placement of the vertices in  $\alpha_4$ .

If we remove the interior of  $\alpha_n$  we obtain a cellular subdivision of  $S^{n-1}$ . It is for this reason that we prefer to speak in this section of  $S^{n-1}$  rather than  $S^n$ . Since every proper subset of the  $(n+1)$  vertices of  $\alpha_n$  span a cell of  $S^{n-1}$ , we see that, for this polytope, every edge is incident with precisely  $(n-1)$  faces, so that we may apply Theorem 2 with  $q = n - 1$ . Since for this polytope, with  $n \geq 3$ ,

$$V = \binom{n+1}{1}, \quad E = \binom{n+1}{2}, \quad F = \binom{n+1}{3}, \quad (3.2)$$

we have, from Theorem 2,

**COROLLARY 1.** *Let  $P_1$  be the polytope obtained by subdividing  $S^{n-1}$  as the boundary of an  $n$ -simplex,  $n \geq 3$ . Then*

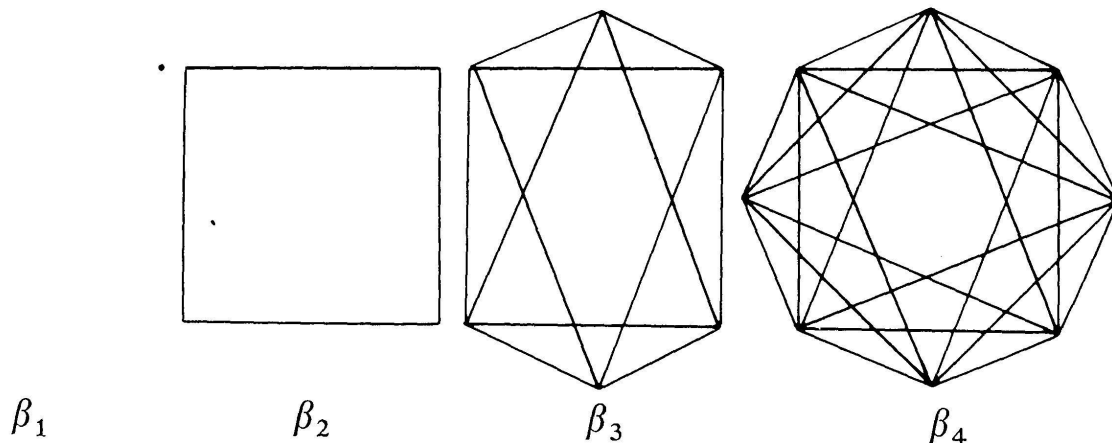
$$\Delta(P_1) = -\frac{\pi}{6}(n-4)(n+1)(n+3).$$

*Proof.* We have, from (3.1) and (3.2)

$$\begin{aligned} \Delta(P_1) &= \pi \left( 2(n+1) - (n-1) \frac{n(n+1)}{2} + 2 \frac{(n+1)n(n-1)}{6} \right) \\ &= \frac{\pi}{6} (n+1) (12 - 3n(n-1) + 2n(n-1)) \\ &= -\frac{\pi}{6} (n+1) (n^2 - n - 12). \end{aligned}$$

It is interesting to note that, while a simplex is convex,  $\Delta(P_1)$  is negative for  $n \geq 5$  (and zero for  $n = 4$ ).

We now turn to our second example of a polytope.



Cross Polytopes

FIGURE 6

CROSS POLYTOPES may be introduced by recognizing that an important aspect of  $n$ -dimensional space is the possibility of having  $n$  mutually perpendicular lines through any point 0. For example, each regular simplex  $\alpha_{n-1}$  (of Figure 5) involves  $n$  points equidistant from 0. Now if we choose to take points equidistant from 0 in both directions we obtain the cellular subdivision of the  $(n-1)$ -sphere called a *cross polytope*. These have  $2n$  vertices and their  $(n-1)$ -cells consist of  $2^n$  of the  $\alpha_{n-1}$ 's. Figure 6 illustrates the cases where  $n$  is equal to 1, 2, 3, and 4 respectively. Thus  $\beta_1$  is a pair of points (vertices) and we can think of progressing from  $\beta_i$  to  $\beta_{i+1}$  by beginning with  $\beta_i$ , introducing a pair of diametrically opposed points (vertices), not in the linear space spanned by  $\beta_i$ , and then joining each of these points to the existing points of  $\beta_i$  (but not to each other). The polytope  $\beta_n$ , which we will call  $P_2$ , is, in fact, homeomorphic to  $S^{n-1}$ . It can easily be shown by induction that

$$N(P_2)_k = 2^{k+1} \binom{n}{k+1}, \quad 0 \leq k \leq n-1.$$

Thus, in particular, for this polyhedron  $P_2$ ,

$$V = 2n, \quad E = 2n(n-1), \quad F = \frac{4}{3}n(n-1)(n-2) \quad (3.3)$$

We now prove

**PROPOSITION 1.** *In the polytope  $P_2$  every edge is incident with precisely  $(2n-4)$  faces,  $n \geq 2$ .*

*Proof.* We first assert that it is plain that in  $\beta_n$  every vertex is incident with precisely  $(2n-2)$  edges. This follows by an easy induction on  $n$ . For  $\beta_{n-1}$  has  $(2n-2)$  vertices and every vertex is, by induction, incident with  $(2n-4)$  edges. Thus a vertex of  $\beta_{n-1}$  is incident with  $((2n-4) + 2)$  edges of  $\beta_n$ , while a new vertex of  $\beta_n$  is incident with  $(2n-2)$  edges of  $\beta_n$ .

Now suppose that, in  $\beta_{n-1}$ , every edge is incident with  $(2n-6)$  faces—this is certainly true if  $n = 3$ . Then an edge of  $\beta_{n-1}$  is incident with  $((2n-6) + 2)$  faces of  $\beta_n$ , while a new edge of  $\beta_n$  is incident with  $(2n-4)$  faces of  $\beta_n$  (since a vertex of  $\beta_{n-1}$  is incident with  $(2n-4)$  edges of  $\beta_{n-1}$ ).

This proof illustrates how we pass from  $\beta_{n-1}$  to  $\beta_n$  by “stepping up dimensions by 1”. This is the point of view of topologists, who introduced such an idea into combinatorial topology without, perhaps, realizing that it had already been introduced by the geometers. Topologists call the passage from  $\beta_{n-1}$  to  $\beta_n$

suspension, and apply this idea to arbitrary topological spaces. Thus the suspension of  $X$  is obtained by joining  $X$  to two independent points or, equivalently, by taking two cones with base  $X$  and joining them together along their bases.

Returning to  $P_2$ , we are now ready to prove

**COROLLARY 2.** *Let  $P_2$  be the polytope obtained by subdividing  $S^{n-1}$  as a cross polytope. Then*

$$\Delta(P_2) = -\frac{4\pi}{3}n(n^2 - 3n - 1).$$

*Proof.* We assemble the facts from (3.1), (3.3) and Proposition 1 to infer that

$$\begin{aligned}\Delta(P_2) &= \pi(4n - 4n(n-1)(n-2) + \frac{8}{3}n(n-1)(n-2)) \\ &= \frac{4\pi n}{3}(3 - (n-1)(n-2)) \\ &= -\frac{4\pi}{3}n(n^2 - 3n - 1).\end{aligned}$$

Here we note that  $\Delta(P_2)$  is negative for  $n \geq 4$ .

Finally we turn to our third example of a polytope.

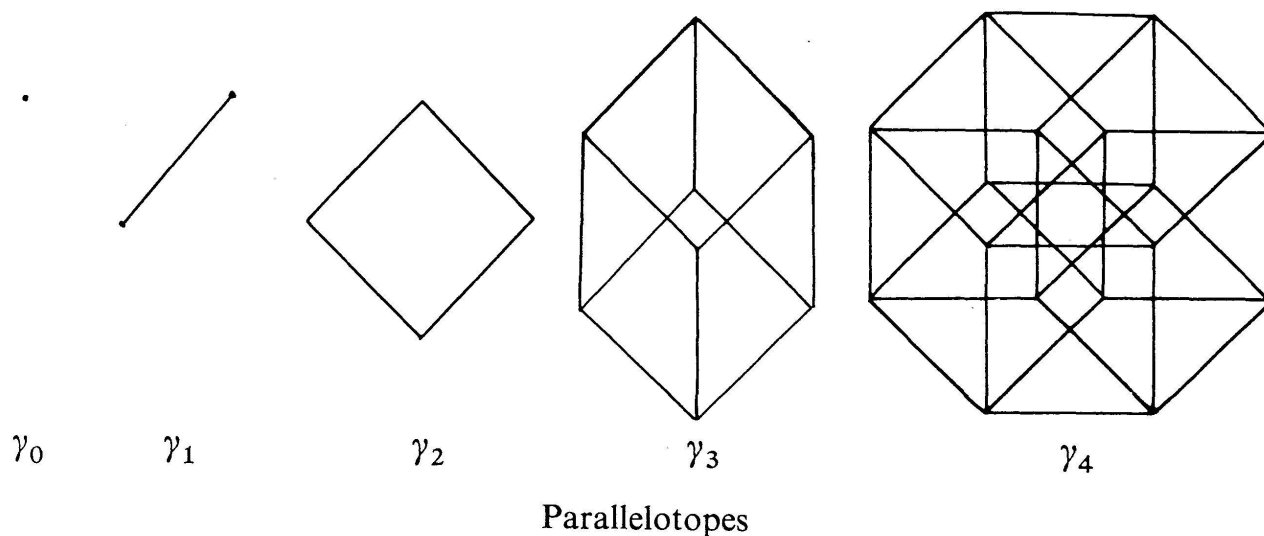


FIGURE 7

PARALLELOTOPES are illustrated in Figure 7. The passage from  $\gamma_i$  to  $\gamma_{i+1}$  is achieved by translating  $\gamma_i$  (not along any of its own lines) from its initial to a final position and then joining in pairs each of the original points with the

corresponding point of the translated figure. If all edges have the same length the polytope is called a *measure polytope*. The quantities  $N_k$  can be computed by considering the passage from  $\gamma_i$  to  $\gamma_{i+1}$ . Thus we readily obtain the inductive relation

$$N(\gamma_{i+1})_k = 2N(\gamma_i)_k + N(\gamma_i)_{k-1}, \quad k \leq i. \quad (3.4)$$

Now  $\gamma_n$  is, combinatorially, a hypercube—we specialize the construction by taking  $\gamma_1$  to be the unit interval and always translating orthogonally by unit distance. Thus the boundary of  $\gamma_n$  is topologically  $S^{n-1}$ . We call the boundary  $P_3$  and infer from (3.4) that

$$N(P_3)_k = 2^{n-k} \binom{n}{k}, \quad 0 \leq k \leq n-1.$$

Note that, for  $n = 3$ , we get, combinatorially, the unit cube, with 8 vertices, 12 edges, and 6 faces. In general the polytope  $P_3$ , with  $n \geq 3$ , yields the values

$$V = 2^n, \quad E = 2^{n-1} n, \quad F = 2^{n-3} n(n-1). \quad (3.5)$$

By an argument very similar to, but simpler than, that of Proposition 1, we may show

**PROPOSITION 3.** *In the polytope  $P_3$ , with  $n \geq 3$ , every edge is incident with  $(n-1)$  faces.*

We are now ready to prove

**COROLLARY 4.** *Let  $P_3$  be the polytope obtained by subdividing  $S^{n-1}$  as the boundary of an  $n$ -dimensional parallelotope,  $n \geq 3$ . Then*

$$\Delta(P_3) = -2^{n-2} \pi (n^2 - n - 8).$$

*Proof.* From (3.1), (3.5) and Proposition 3 we have

$$\begin{aligned} \Delta(P_3) &= \pi (2^{n+1} - 2^{n-1} n(n-1) + 2^{n-2} n(n-1)) \\ &= 2^{n-2} \pi (8 - n(n-1)) \\ &= -2^{n-2} \pi (n^2 - n - 8). \end{aligned}$$

Here we note that  $\Delta(P_3)$  is negative for  $n \geq 4$ .

The fact that  $\Delta(P_1)$ ,  $\Delta(P_2)$ , and  $\Delta(P_3)$  are all different (except for  $n = 3$ ) shows that the total angular deficiency has no chance of being a topological

invariant for polyhedra of dimension  $\geq 3$ . On the other hand it is still striking that  $\Delta$  depends only on the cellular structure and is independent of the underlying geometric structure.

*Remarks.* (a) The polytopes  $P_1, P_2, P_3$  not only enjoy the property that each edge of  $P_i$  is incident with the same number of faces of  $P_i, i = 1, 2, 3$ —the property we used to calculate  $\Delta(P_i)$  from Theorem 1—they also enjoy the property that each face has the same number of sides. This latter property could also have been used to calculate  $\Delta(P)$ . Thus if  $P$  is a polyhedron subdivided so that each face has the same number  $s$  of sides, then one may show that

$$\Delta(P) = 2\pi V - \pi F(s-2). \quad (3.6)$$

It is easy to deduce either of the formulae (3.1), (3.6) from the other if the polyhedron  $P$  enjoys *both* the relevant properties. For if every edge of  $P$  is incident with  $q$  faces and every face of  $P$  has  $s$  sides, then

$$qE = sF. \quad (3.7)$$

Of course there is an equality corresponding to (3.7) in higher dimensions.

(b) The polytopes  $P_2$  and  $P_3$  may be regarded as *dual*, in the sense that there is a one-one correspondence between the cells of  $P_2$  of dimension  $k$  and the cells

FIGURE 8

Data for  $P_1, P_2, P_3$  when  $n = 4$ 

Name of polytope ( $P_i$ )	$N_0$	$N_1$	$N_2$	$N_3$	Number of sides on each $N_2$	Number of faces incident with each edge	$\Delta$
Simplex ( $P_1$ )	5	10	10	5	3	3	0
Cross polytope ( $P_2$ )	8	24	32	16	3	4	$-16\pi$
Parallelotope ( $P_3$ )	16	32	24	8	4	3	$-16\pi$



of  $P_3$  of dimension <sup>1)</sup>  $(n-1) - k$ . Moreover, the incidence relations are carried over by this duality; thus if, in  $P_2$ , every  $(i-1)$ -face is incident with <sup>2)</sup>  $s_i$   $i$ -faces, then, in  $P_3$ , every  $(n-i)$ -face is incident with  $s_i$   $(n-i-1)$ -faces (and there is a symmetrical statement interchanging  $P_2$  and  $P_3$ ). In this sense  $P_1$  is selfdual. Figure 8 displays these dualities for  $n = 4$ , as well as the value of  $\Delta$ .

### 3. HISTORICAL COMMENT AND SUMMARY

René Descartes (1596–1650) and Leonhard Euler (1707–1783) worked on these subjects independently—yet, as we have seen, Pólya (1887– ) has shown that their seemingly different formulae for convex polyhedra homeomorphic to  $S^2$  are entirely equivalent to each other. One might believe from the evidence that Euler may have known about Descartes' work on this subject. That would be an erroneous assumption since Descartes' work on this matter [5] was not printed until a century after Euler's death (see [1], p. 56).

Euler [6] offered a variety of verifications but no formal proof of his formula. We have observed that each of the formulae is somewhat surprising by itself and that their connection rather defies intuition since at first glance they seem to be dealing with different qualitative aspects of polyhedra. As a matter of fact neither Euler's nor Descartes' formula is easy to *prove* independently; yet, as we have seen, it is not at all difficult to follow Pólya's proof that the two formulae are equivalent.

The formulae diverge in higher dimensions so that their relationship is a special phenomenon of dimension 2. Euler's formula was generalized by Ludwig Schläfli [9], a Swiss mathematician of the 19th century (1814–1895), who described, in effect, the Euler-Poincaré characteristic of an  $n$ -dimensional sphere  $S^n$ , subdivided as a *polytope*, a combinatorial structure attributed by Coxeter to Reinhold Hoppe [11]. Poincaré (1854–1912) gave a definition of the Euler-Poincaré characteristic for arbitrary polyhedra, and one proves now, by invoking the topological invariance of the homology groups (see [12]) that the Euler-Poincaré characteristic is a topological invariant.

<sup>1)</sup> The precise form of this duality shows how "correct" it is to regard  $S^{n-1}$  as  $(n-1)$ -dimensional, rather than  $n$ -dimensional.

<sup>2)</sup> In fact,  $s_i = 2(n-i-1)$ .