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SCHUBERT CALCULUS OF A COXETER GROUP

by Howard L. HILLER [1])

INTRODUCTION

Let (W, S) be a finite Coxeter system, in the sense of Bourbaki [6]. It is well-known that W can be realized as the Weyl group of a (possibly non-crystallographic) root system Δ in a real Euclidean space V of dimension $n = |S|$. This space possesses a basis Σ of simple roots such that the reflections s_α through the hyperplane perpendicular to $\alpha \in \Sigma$ precisely yield the generating set S. In this fashion, W admits a natural representation on V, so we can make it act on the polynomial algebra $S(V)$ on V by $w . f(x) = f(w^{-1}x)$. The invariant subalgebra splits up into its homogenous components $S(V)^W = \bigoplus_{j=0}^{\infty} S_j(V)^W$ and the positive components generate a graded homogenous ideal I_W. We can form the quotient algebra $S_W = S(V)/I_W$ which we refer to as the *coinvariant algebra* of W. Of course, Chevalley's theorem [8] tells us that $S(V)^W$ has n algebraically independent generators whose degrees $d_1, \ldots, d_n$ (the fundamental degrees) are useful in describing the gross structure of S_W. In particular, one can compute the Poincaré series of $S_W = \bigoplus_j S_{W,j}$

$$PS(S_W, t) = \sum_{j=0}^{\infty} \dim_{\mathbf{R}} (S_{W,j})\, t^j = \prod_{i=1}^{n} (1 + t + \ldots + t^{d_i - 1})$$

so that the real dimension is $PS(S_W, 1) = \prod_{i=1}^{n} d_i = |W|$ and $S_{W,j} = 0$, $j > \deg(PS(S_{W,t})) = \sum_{i=1}^{n} (d_i - 1)$. Note that the last sum is also equal to the number N of reflections in W, for example, by a formula of Solomon [19].

We are interested here in a finer analysis of the algebraic and W-module structure of S_W. Following Demazure [11], we describe a sort of algebraic

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Bruhat decomposition for S_W relative to a root system for W (section 2). By this we mean an algebra H_W is constructed with a basis $\{X_w\}_{w \in W}$ and a map $c: S(V) \to H_W$, that induces an isomorphism $S_W \approx H_W$. The basis depends on the relative lengths of the simple roots. In the case where W is a Weyl group, and the lengths of the roots are chosen to make the Cartan matrix integral, the element X_w corresponds to the cocyle dual to the Ehresmann-Bruhat cell decomposition of a certain flag manifold $G/B = K/T$. Hence, for example, the Coxeter group $\Sigma_n \int \mathbf{Z}_2$ admits the two different Schubert calculi of type B_n and C_n. In addition, the map c above corresponds to taking the first Chern class of the line bundle associated to a character of T (where V is thought of as the character group $X(T)$ on the maximal torus). Our first task is to describe a section for the map c (section 3). We think of this as a Giambelli formula for S_W. This leads us to introduce a notion of fundamental weights for a Coxeter system, which turns out to yield the 1-dimensional generators X_{s_α}, $\alpha \in \Sigma$. This allows us to view an arbitrary X_w as a polynomial in the X_{s_α}'s.

In section 4, we look closer at the multiplicative structure of H_W. By our Giambelli formula, it suffices to understand multiplication of X_w by a fundamental weight. Here we exploit a commutator computation of Bernstein, Gelfand and Gelfand [2] to get such a Pieri formula.

It is possible to relativize the above results. In section 5 we recall the basic facts about the lattice of parabolic subgroups $\{W_\theta\}_{\theta \subset S}$ of the Coxeter group W. Of course, (W_θ, θ) is a Coxeter system itself. We consider the invariant algebra $H_W^{W_\theta}$ and show that it is generated by $\{X_w\}_{w \in W^\theta}$ where W^θ is a familiar set of coset representatives for W_θ in W.

Finally, using the results of section 5 and the parabolic $\Sigma_k \times \Sigma_n \subset \Sigma_{n+k} = W(A_{n+k-1})$ ge give an algebraic derivation of the classical Pieri formula of the Schubert calculus of a complex grassmannian.

Section 1 is a brief review of facts about Coxeter groups we will require in the sequel.

We note in passing that the sort of results described here have already been analyzed from a variety of viewpoints—the Chow ring [12], Lie algebra cohomology [17], and De Rham cohomology [22], to mention a few. The advantage of our method, inspired by [2] and [11] is that once one has identified the algebra in question as the coinvariant algebra S_W, all of the Schubert machinery follows in a purely formal fashion.

It is hoped that an extension of this circle of ideas to affine Weyl groups will shed some light on the Bott decomposition of the loop space of a Lie group [15].

1. Coxeter groups

We begin by reviewing some of the elementary theory of Coxeter groups. Some detail is included to avoid any oblique use of the crystallographic condition. Following Bourbaki [6, IV] we say (W, S) is a *finite Coxeter system* if W is a finite group given by the presentation $\langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where m_{ij} is the order of $s_i s_j$. It is possible [6, V] to construct a real Euclidean space V and a root system (Δ, Σ) in V that "geometrically realizes" (W, S). By this we mean the following. If $\gamma \in \Delta$ then

$$s_\gamma(x) = x - (x, \gamma^v)\gamma \qquad \left(\text{co-root} \quad \gamma^v = \frac{2\gamma}{(\gamma, \gamma)}\right)$$

is the reflection through the hyperplane perpendicular to the root γ, and we can form the subgroup $W(\Delta)$ of $GL(V)$ generated by the s_γ's, $\gamma \in \Delta$. In fact, the s_α's, $\alpha \in \Sigma$, generate $W(\Delta)$ and we call the pair $(W(\Delta), \{s_\alpha : \alpha \in \Sigma\})$ the *Weyl system* of (Δ, Σ). Coxeter [9] proved that the Weyl system is always a Coxeter system and if this pair is isomorphic (in the obvious sense) to (W, S) we say (Δ, Σ) is a geometric realization of (W, S). Of course, the choice of such a (Δ, Σ) is not unique. But clearly up to a rigid motion of V, the root system is determined by the lengths of the simple roots.

If the lengths can be chosen so that $(\alpha, \beta^v) \in \mathbf{Z}$ for all $\alpha, \beta \in \Sigma$, we say W is *crystallographic* (or a Weyl group). Geometrically, this means that the $\mathbf{Z}$-lattice generated by Σ is preserved by W. As mentioned in the introduction, even this choice of relative lengths is not necessarily unique.

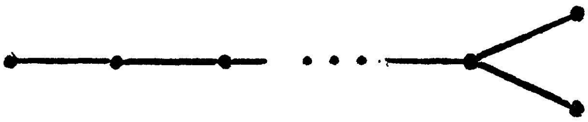
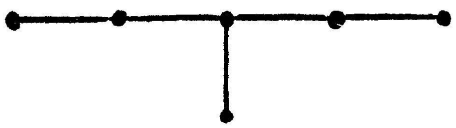
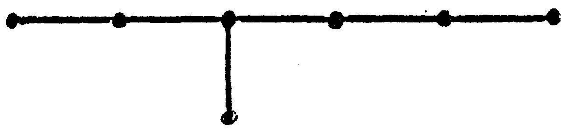
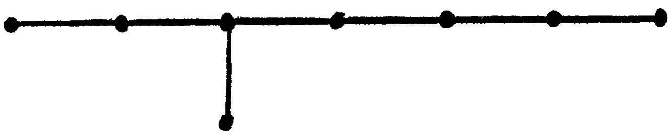
We can choose a vector $t \in V$, such that $(t, \alpha) > 0$ for all $\alpha \in \Sigma$ (i.e. t is in the fundamental chamber C). This vector decomposes the roots $\Delta = \Delta^+ \coprod \Delta^-$ where

$$\Delta^+ = \{\gamma \in \Delta : (\gamma, t) > 0\}$$

and $\Delta^- = -\Delta^+$. Note that $|\Delta^+| = N = \frac{1}{2}|\Delta|$, where N is the number of reflections in W as described in the introduction.

It is now customary to attach an edge labelled graph to (W, S) called the Coxeter graph. The nodes correspond to the elements of S and s_i is attached to s_j by an edge if $m_{ij} \geqslant 3$, and if also $m_{ij} > 3$ the edge is labelled with the number m_{ij}. In 1934, Coxeter [9] classified the Coxeter groups with connected graphs and showed that every Coxeter group is a product of the "connected" components. The classification of the irreducible Coxeter groups along with the fundamental degrees is

TABLE

W	Coxeter graph	$d_1, \ldots, d_n$
A_n		2, 3, ..., $n+1$
B_n		2, 4, ..., $2n$
D_n		2, 4, ..., $2(n-2)$, $2(n-1)$, n
E_6		2, 5, 6, 8, 9, 12
E_7		2, 6, 8, 10, 12, 14, 18
E_8		2, 8, 12, 14, 18, 20, 24, 30
F_4		2, 6, 8, 12
G_2		2, 6
H_3		2, 6, 10
H_4		2, 12, 20, 30
$I_2(m)$		2, m

We will assume throughout that W is irreducible.

The crystallographic Coxeter groups and their root systems are well-known and correspond up to a choice of relative lengths of the simple roots to the Cartan classification of simple Lie algebras over the complex numbers. The dihedral groups are the Weyl groups of $I_2(m)$, and are the symmetry groups of a regular m-gon (from which it is easy to construct (Δ, Σ)). The group H_3 is isomorphic to a product of $\mathbf{Z}_2$ and an alternating group on five letters and H_4 is the symmetry group of a certain 4-dimensional polytope [9, 10].

The primary piece of structure available on a Coxeter group is the *length function* $l : W \to \mathbf{N}$, where $l(w)$ is defined as the minimal length of an expression of w in the generators S. If $l(w) = k$ and $w = s_1 \dots s_k$, $s_i \in S$, we call this a *reduced decomposition* of W. There is an alternative intrinsic description.

LEMMA 1.1. *Let* Γ_w *denote the set of* $\gamma \in \Delta^+$ *such that* $w(\gamma) \in \Delta^-$, *then*

(i) $|\Gamma_{ws_\alpha}| = |\Gamma_w| \pm 1$ *if and only if* $w(\alpha) \in \Delta^{\pm}$,

(ii) $l(w) = |\Gamma_w|$,

(iii) $l(ws_\alpha) = l(w) \pm 1$ *if and only if* $w(\alpha) \in \Delta^{\pm}$.

Proof. To see (i) one need only recall that $\Gamma_{s_\alpha} = \{\alpha\}$. This first assertion then implies $|\Gamma_w| \leqslant l(w)$. The other inequality follows from an induction on $|\Gamma_w|$ and then (ii) follows. (iii) is immediate from (i) and (ii).

The next piece of structure on the Coxeter group we require is the so-called Bruhat ordering [13]. We define $w' \to w$ (intuitively, w' is an immediate predecessor of w if there exists a positive root γ such that $\sigma_\gamma w = w'$ and $l(w') = l(w) + 1$. (We will often adorn $\to$ with the unique such γ.) Since W is transitive on the roots and $ws_\alpha w^{-1} = s_{w(\alpha)}$ the first condition is equivalent to $w' w^{-1}$ being a conjugate of a fundamental reflection $s \in S$. The *Bruhat order* $<$ on W is the transitive closure of the ordering $\to$. Note that l is forced to be strictly order-preserving so that the two pieces of structure we have introduced are compatible. We can now relate $\to$ to any particular reduced decomposition of w.

LEMMA 1.2. *If* $w = s_1 \dots s_k$ *is a reduced decomposition, then* $w' \to w$ *only if* $w' = w_{\hat{i}}$ *where* $w_{\hat{i}} = s_1 \dots \hat{s_i} \dots s_k$ *(and* $\hat{}$ *denotes deletion).*

Proof. See Theorem 1.1 (III) in [13].

Hence, in general, the Bruhat ordering corresponds to the subwords of any reduced decomposition. So, for any i we can find a $\gamma \in \Delta^+$ such that $s_\gamma w_{\hat{i}} = w$. The next result describes these roots γ both specifically and abstractly.

LEMMA 1.3. *If* $w = s_1 \dots s_k$ *is a reduced decomposition, define* $\theta_i = s_1 \dots s_{i-1}(\alpha_i)$ *where* $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$. *Then the following sets are equal*

(i) $\Gamma_{w^{-1}} = \Delta^+ \cap w(\Delta^-)$,

(ii) $\{\theta_i\}_{1 \leq i \leq k}$,

(iii) $\{\gamma \in \Delta^+ : s_\gamma \hat{w}_i = w\}$.

Proof. (i) $\subseteq$ (ii). Let $\gamma \in \Delta^+$ and $w^{-1}(\gamma) \in \Delta^-$. Let j be the smallest number such that $s_j \dots s_1(\gamma) \in \Delta^-$. Then $\alpha_j = s_{j-1} \dots s_1(\gamma)$. Hence $\gamma = \theta_j$.

(ii) $\subseteq$ (iii). It suffices to compute

$$\begin{aligned} s_{\theta_i} \hat{w}_i &= s_{s_1 \dots s_{i-1}(\alpha_i)} (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1 (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_k = w . \end{aligned}$$

But now $|\Gamma_{w^{-1}}| = l(w^{-1}) = l(w) = k$, by (1.1) and certainly $|\{\gamma \in \Delta^+ : s_\gamma \hat{w}_i = w\}| \leqslant k$, so all three sets must be equal.

Remark. Though the θ_i's are defined in terms of a reduced decomposition, (1.3 i) shows that they are actually independent of the choice made.

We now recall that the Bruhat order on W possesses a unique top element of greatest length.

LEMMA 1.4. *There exist a unique element* $w_0 \in W$ *such that* $l(w_0) = N$. *In addition,* $w_0 \geqslant w$, *for all* $w \in W$, $w_0^2 = 1$ *and* $l(ww_0) = l(w_0) - l(w)$.

Proof. One knows that W acts simply transitively on the chambers and w_0 is chosen to be the unique element satisfying $w_0 C = -C$. The rest is standard, see [6, p. 43].

Finally, we make some remarks on the (anti) invariant theory of Coxeter groups. The main result is

PROPOSITION 1.5. *If* (W, S) *is a Coxeter system, then the invariant algebra* $S(V)^W$ *has* $|S|$ *algebraically independent generators of degrees* $2 = d_1, d_2, \dots, d_n$. *Equivalently,* $S(V)$ *is a free* $S(V)^W$*-module.*

Proof. This follows immediately from Chevalley's theorem [8].

Remark. It is often useful in this context to think of W as the Galois group of the rational function field $\overline{S(V)}$ over the rational function field $\overline{S(V)^W}$ of the invariants. We exploit this point of view in the next section.

There is also a theory of anti-invariants, i.e. polynomials $u \in S(V)$ such that $w \cdot u = (-1)^{l(w)} u$. The algebra of anti-invariants is written $S(V)^{-W}$. It is a free module of rank 1 over $S(V)^W$ generated by the element $d = \prod_{\gamma \in \Delta +} \gamma \in S_N(V)$. The corresponding "anti-averaging" operating is $\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u$.

2. Demazure's basis theorem

Let $\varepsilon : S(V) \to S_0(V) \approx \mathbf{R}$ denote the projection map. We begin by defining certain operators on $S(V)$, whose composition with ε should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let (W, S) be a Coxeter system and (Δ, Σ) a geometric realization of it.

Definition 2.1. If $\alpha \in \Delta$, define $\Delta_\alpha = \alpha^{-1}(1-s_\alpha)$ as an $S(V)^W$-endomorphism of $S(V)$. (Note the division is legitimate since s_α is the identity on the $\ker(\alpha) = \alpha^\perp$; thinking of α as a linear form $x \mapsto (x, \alpha)$ in $V^* = S_1(V)$, of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

Lemma 2.2. *If* $w \in W, \alpha \in \Delta, u, v \in S(V)$ *then*

(i) $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)}$,

(ii) $\Delta_\alpha^2 = 0$,

(iii) $s_\alpha = 1 - \alpha \Delta_\alpha$,

(iv) $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$ *(where the superscript denotes invariants)*

(v) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v)$,

(vi) $\Delta_\alpha(I_W) \subset I_W$,

(vii) $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^v, \omega) s_\alpha$,

where ω^* *denotes the operator multiplication by* ω.

We now define $\mathbb{A}_W$ to be the subalgebra of the algebra of endomorphisms $\mathrm{End}(S(V))$ generated by the Δ_α's ($\alpha \in \Delta$) and ω^*, $\omega \in S(V)$. Note Δ_α decreases the grading by (-1) and $W \subseteq \mathbb{A}_W$ by (2.2 iii).

There is a map $\varepsilon_* : \text{End}\, S(V) \xrightarrow{i} S(V)^*$ obtained by composition with ε and we write $\overline{\mathbb{A}}_W = \varepsilon_* \mathbb{A}_W \subseteq S(V)^*$. Double duality over **R** gives a map

$$c : S(V) \xrightarrow{\delta} S(V)^{**} \xrightarrow{i*} \overline{\mathbb{A}}_W^* .$$

We will write $H_W = \bar{\Delta}_W^*$, christened by Demazure, the cohomology ring of (Δ, Σ) and c the characteristic homomorphism. Demazure [11, Prop. 2] makes the basic observation that c induces a unique graded algebra and W-module structure on H_W compatible with $S(V)$. (We should mention here that H_W depends on the lengths of the simple roots though the notation obscures this.) The first task is to extend the class of operators $\Delta_\alpha = \Delta_{s_\alpha}$ from S to the entire Coxeter group W. Naturally, we will define $\Delta_w = \Delta_{\alpha_1} \ldots \Delta_{\alpha_k}$ where $w = s_{\alpha_1} \ldots s_{\alpha_k}$ is a reduced decomposition of w, once we have proven that this definition is independent of the choice of the decomposition. Our information on Coxeter groups is a possible route but instead we follow Demazure's argument since it leads to worthwhile dividends. We begin with a few lemmas.

LEMMA 2.3: *Let d denote $\prod_{\alpha \in \Delta +} \alpha \in S_N(V)$. If $w_0 = s_{\alpha_1} \ldots s_{\alpha_N}$ is the longest word then*

$$\Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_N} = d^{-1}((-1)^N w_0 + \sum_{w \neq w_0} q_w w)$$

where $q_w \in \overline{S(V)}$.

Proof. We compute

$$\begin{aligned} \Delta_{\alpha_1} \ldots \Delta_{\alpha_N} &= \alpha_1^{-1}(1 - s_{\alpha_1}) \ldots \alpha_N^{-1}(1 - s_{\alpha_N}) \\ &= (-1)^N \alpha_1^{-1} s_1 \ldots \alpha_1^{-1} s_N + \sum_{w \neq w_0} a(w)\, w \end{aligned}$$

where the index of summation in the last term is a consequence of (1.4). It now suffices to watch in the first term what happens to inverted roots α_i^{-1} as we pass the fundamental reflections s_i over to the right. Using (2.2 i) we get

$$(-1)^N \Big(\prod_{i=1}^{n} s_{\alpha_1} \ldots s_{\alpha_{i-1}}(\alpha_i) \Big)^{-1} s_{\alpha_1} \ldots s_{\alpha_N}$$

But by (1.3) this is $(-1)^N d^{-1} w_0$ since $w_0^{-1} = w_0$ converts all positive roots into negative roots by (1.4). We now let $q_w = da_w$ and the proof is complete.

LEMMA 2.4. *If* $f \in \operatorname{End}(S(V))$ *reduces the grading by* N *then* $d \cdot f = \lambda J$ *for some* $\lambda \in \mathbf{R}$, *where* $J = \sum_{w \in W} (-1)^{l(w)} w$.

Proof. See [11, Prop. 1 (b)].

PROPOSITION 2.5: *If* $w_0 = s_{\alpha 1} \dots s_{\alpha N}$ *as above, then* $\Delta_{\alpha 1} \dots \Delta_{\alpha N} = d^{-1} J$.

Proof. By (2.4), $d\Delta_{\alpha_1} \dots \Delta_{\alpha N} = \lambda J = \lambda(-1)^N w_0 + \sum_{w \neq w_0} (-1)^{l(w)} \lambda w$. Also by (2.3) $d\Delta_{\alpha_1} \dots \Delta_{\alpha N} = (-1)^N w_0 + \sum_{w \neq w_0} q_w w$. By Dedekind's theorem (see, e.g., [1]) the w's are independent as automorphisms of $\overline{S(V)}$, so $\lambda = 1$ and the result follows.

We can now show

PROPOSITION 2.6. Δ_w *is well defined.*

Proof. By [6, IV § 1, Prop. 5], it suffices to show

$$\Delta_\alpha \Delta_\beta \Delta_\alpha \dots = \Delta_\beta \Delta_\alpha \Delta_\beta \dots$$

with $m_{\alpha\beta}$ terms on each side. But the dihedral root system $I_2(m_{\alpha\beta})$ or $A_1 \times A_1$ has $s_\alpha s_\beta s_\alpha \dots = s_\beta s_\alpha s_\beta \dots$ as its longest word and hence (2.5) completes the argument.

THEOREM 2.7. *The* $\{\Delta_w\}_{w \in W}$ *are an* $S(V)$*-basis for* $\mathbb{\Delta}_W$ *and hence the* $\{\varepsilon \circ \Delta_w\}_{w \in W}$ *are an* **R***-basis for* $\overline{\mathbb{\Delta}}_W$.

Proof. By (2.2 v), it is easy to check the Δ_w'x generate $\mathbb{\Delta}_W$ as an $S(V)$-module. The linear independence follows from Dedekind's theorem, and the last statement is immediate.

We now define $\{X_w\}_{w \in W}$ to be the basis of $H_W = \overline{\mathbb{\Delta}}{}_W^*$ dual to the basis $\{\varepsilon \cdot \Delta_w\}_{w \in W}$ of $\overline{\mathbb{\Delta}}_W$, i.e.

$$X_w(\varepsilon \cdot \Delta_{w'}) = \delta_{ww'}$$

This immediately yields the following "coordinate-wise" description of c.

$$c(u) = \sum_{w \in W} \varepsilon \Delta_w(u) X_w .$$

First, we show c has the correct kernel. We need the following Lemma that follows from R. Steinberg [21].

LEMMA 2.8. *If* I *is a graded ideal of* $S(V)$ *such that* $I_W \subseteq I$ *and* $\mathbf{R}d \cap I = 0$, *then* $I = I_W$.

This result rapidly yields our version of the "basis theorem" of the Schubert calculus, namely

THEOREM 2.9. Ker $(c) = I_W$ *and* c *induces an isomorphism* $S_W \approx H_W$.

Proof. For the first assertion, by (2.8), it suffices to compute

$$\begin{aligned} c(\lambda d) &= \lambda \sum \varepsilon \Delta_w(d) X_w = \lambda \Delta_{w_0}(d) X_{w_0} \\ &= \lambda \,|\, W \,|\, X_{w_0} \,. \end{aligned}$$

Finally, c is clearly onto by construction.

In the next section we will work on producing an explicit section for c.

Remark. Demazure's proof, though restricted to Weyl groups, is done integrally. In that situation, c is not onto, and Demazure computes the order of the finite quotient. It corresponds to the usual notion of torsion in Lie groups [3, 5]. Indeed, the point is that only when W preserves some integral lattice can one hope to carry out an analysis in integral cohomology; in the general case we must resort to real cohomology, as we do here. Of course, the torsion problems then disappear.

3. GIAMBELLI FORMULA

We begin with an easy lemma.

LEMMA 3.1. Δ *is quasi-multiplicative, i.e.*

$$\Delta_w \cdot \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first clause is immediate since the conditions implies that reduced decompositions of w and w' can be juxtaposed to yield a reduced decomposition of ww'. Now suppose $w = s_\alpha w'$ and $l(s_\alpha w') = l(w') - 1$ (that this is the only possibility that follows from (1.1)). Then $w' = s_\alpha (s_\alpha w')$ and

$$l(w') = 1 + \big(l(w') - 1\big) = l(s_\alpha) + l(s_\alpha w')$$

so by the first part $\Delta_{w'} = \Delta_{s_\alpha} \Delta_{s_\alpha w'}$. But

$$0 = \Delta_{s_\alpha} \Delta_{s_\alpha} \Delta_{s_\alpha w'} = \Delta_{s_\alpha} \Delta_{w'}$$

by (2.2 ii) and induction on $l(w)$ completes the proof.

COROLLARY 3.2. $\varepsilon \cdot \Delta_{w'} \Delta_{w^{-1}w_0} = \delta_{ww'} \Delta_{w_0}$ *on* $S_N(V)$.

Proof. If $w' = w$, then by (1.4) and (3.1)

$$\Delta_w \Delta_{w^{-1}w_0} = \Delta_{w_0}$$

and the result follows.

We now need only consider $w' \neq w$, but with $l(w) = l(w')$, (otherwise, we are done for dimensional reasons). Thus

$$l(w') + l(w^{-1}w_0) = l(w') + (l(w_0) - l(w)) = l(w_0)$$

and

$$l(w'w^{-1}w_0) = l(w_0) - l(w'w^{-1}) \neq l(w_0)$$

So by (3.1), $\Delta_{w'} \Delta_{w^{-1}w_0} = 0$, and the proof is complete.

It is now easy to dualize this to the following assertion:

COROLLARY 3.3 (Giambelli formula). $c\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) = X_w$. *Hence in particular,* $c\left(\frac{d}{|W|}\right) = X_{w_0}$.

Proof.
$$c\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) = \sum_{w' \in W} \varepsilon \Delta_{w'}\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) X_{w'}$$
$$= \sum_{\substack{w' \in W \\ l(w') = l(w)}} \delta_{ww'} \varepsilon \Delta_{w_0}\left(\frac{d}{|W|}\right) X_{w'}$$
$$= X_w \qquad \text{by (2.5).}$$

Note that the map $\sigma : X_w \mapsto \Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)$ is a vector space section for c. In the remainder of this section we will find other I_W-equivalent expressions for X_{w_0} and use these to put σ in a more manageable form. We will call X_{w_0} the *fundamental class* of the cohomology ring H_W.

Example. Let $W = W(A_{n-1}) = \Sigma_n$. As usual, the positive roots Δ^+ are $\{e_i - e_j : i < j\}$ where $\{e_i\}$ is the standard basis of $\mathbf{R}^n$. Hence, the fundamental class is c of a multiple of the Vandermonde determinant, namely

$$\frac{1}{n!}\begin{vmatrix} 1 & e_n & e_n^2 & \dots & e_n^{n-1} \\ 1 & e_{n-1} & e_{n-1}^2 & \dots & e_{n-1}^{n-1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & e_1 & e_1^2 & \dots & e_1^{n-1} \end{vmatrix}$$

In this example we used the standard basis for V. The following result indicates that a Coxeter generalization of the fundamental weight basis is more appropriate in our situation. Recall the *fundamental weights* $\{\omega_\alpha\}_{\alpha\in\Sigma}$ are given by the requirement

$$(\omega_\alpha, \beta^v) = \delta_{\alpha\beta}$$

We now have

LEMMA 3.4.

(i) $\Delta_\beta(\omega_\alpha) = \delta_{\alpha\beta}$,

(ii) $c(\omega_\alpha) = X_{s_\alpha}$,

(iii) $c(\alpha) = \sum_{\beta\in\Sigma} (\alpha, \beta^v) X_{s_\beta}$.

Proof.

(i) $$\Delta_\beta(\omega_\alpha) = \beta^{-1}(\omega_\alpha - s_\beta(\omega_\alpha)) = \beta^{-1}(\omega_\alpha - (\omega_\alpha - (\omega_\alpha, \beta^v)\beta)) = (\omega_\alpha, \beta^v) = \delta_{\alpha\beta}$$

(ii) $$c(\omega_\alpha) = \sum_{w\in W} \varepsilon \Delta_w(\omega_\alpha) X_w = \sum_{\beta\in\Sigma} \Delta_\beta(\omega_\alpha) X_{s_\beta} = X_{s_\alpha}$$

(iii) Since $\alpha = \sum_{\beta\in\Sigma} (\alpha, \beta^v)\, \omega_\beta$, the result follows immediately from (ii).

This result tells us that if we can write X_w as c of some polynomial in the $\{\omega_\alpha\}_{\alpha\in Z}$ or $\{\alpha\}_{\alpha\in Z}$ we will have also written X_w as a polynomial in the X_{s_α}'s. We will often abbreviate the Cartan matrix entries by $c_{\alpha,\beta} = (\alpha, \beta^v) = -\frac{\|\alpha\|}{\|\beta\|} \cos\left(\frac{\pi}{m_{\alpha\beta}}\right)$. In practice, it is maximally efficient to write X_w as a polynomial in the simple roots, since then an easy substitution will yield either a polynomial in the weights or a polynomial in the original coordinate variables $e_1, \ldots, e_n$.

It is possible to relate the fundamental class X_{w_0}, with the invariant theory of W.

PROPOSITION 3.5. *Let* $f_1, \ldots, f_n$ *be fundamental invariants for* W. *Then, if* $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ *is the Jacobian of these polynomials there is a real number* λ *such that*

$$c(\lambda J) = X_{w_0}.$$

Proof. This follows from the stronger, well-known assertion that d divides J [20, p. 85]. (It also follows from the theory of complete intersection rings.)

In the interest of understanding the Giambelli formula (3.3) we deduce some formulae for $\Delta_w(d)$. If $\{\alpha_i\}_{1 \leq i \leq n}$ are distinct positive roots we denote by $d_{\alpha_1, \ldots, \alpha_n}$ the product $d \cdot \prod_{i=1}^{n} \alpha_i^{-1} = \prod_{\substack{\alpha \in \Delta \\ \alpha \neq \alpha_i}} \alpha$. It is easy to see

LEMMA 3.6.

$$s_\beta(d_{\alpha_1, \ldots, \alpha_n}) = \begin{cases} d_{s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n), \beta} & \textit{if } \beta = \alpha_j, \\ -d_{s_\beta(\alpha_1), \ldots, s(\alpha_n)} & \textit{otherwise.} \end{cases}$$

Proof. Since s_β permutes the set $\Delta^+ - \{\beta\}$, it also permutes

$$\Delta^+ - \{\beta, \alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)\},$$

where $\beta \neq \alpha_i$, for all i. Hence

$$\begin{aligned} s_\beta(d_{\alpha_1, \ldots, \alpha_n}) &= s_\beta(d_{\beta, \alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)}) \cdot s_\beta(\beta) \cdot s_\beta^2(\alpha_1) \circ \ldots \circ s_\beta^2(\alpha_n) \\ &= d_{\beta, \alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)} \cdot (-\beta) \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \\ &= -d_{s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)} \end{aligned}$$

Similarly in the other case.

PROPOSITION 3.7.

$$\Delta_\beta(d_{\alpha_1, \ldots, \alpha_n}) = \begin{cases} \displaystyle\sum_{\substack{s \neq \phi \\ s \subseteq \{1, \ldots, \hat{j}, \ldots, n\}}} (-1)^{|s|} \prod_{i \in s} c_{\alpha_i, \beta} \cdot \\ \beta^{|s|-1} d_{\{\alpha_i : i \in s\}, s_\beta(\alpha_1), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_\beta(\alpha_n)} \\ \textit{if } \beta = \alpha_j \\ d_{\alpha_1, \ldots, \alpha_n, \beta} + d_{s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n), \beta} \\ \textit{otherwise} \end{cases}$$

Proof. The second case is easy so we look at the first

$$\begin{aligned} \Delta_\beta(d_{\alpha_1, \ldots, \alpha_n}) &= \beta^{-1}(d_{\alpha_1, \ldots, \alpha_n} - d_{s_\beta(\alpha_1), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_\beta(\alpha_n), \beta}) \\ &= \beta^{-1}\big[d_{\alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_\beta(\alpha_n)} \\ &\quad \cdot (s_\beta(\alpha_1) \cdot \ldots \cdot \widehat{s_\beta(\alpha_j)} \cdot \ldots \cdot s_\beta(\alpha_n) - \alpha_1 \cdot \ldots \cdot \hat{\alpha}_j \cdot \ldots \cdot \alpha_n)\big] \\ &= d_{\alpha_1, \ldots, \alpha_n, s_\beta(\alpha_j), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_{\beta n}(\alpha)} \\ &\quad \beta^{-1}\Big(\prod_{i \neq j} (\alpha_j - (\alpha_j, \beta^v)\beta) - \prod_{i \neq j} \alpha_i\Big) \end{aligned}$$

and after writing the product as a sum the desired expression follows.

It is possible to use (3.7) to explicitly compute polynomial expressions for X_w.

Example. Let $W = W(A_2)$ where A_2 is the root system in $\mathbf{R}^3$ with simple roots $\Sigma = \{\alpha = e_1 - e_2, \beta = e_2 - e_3\}$ and the additional positive root $\alpha + \beta = e_1 - e_3$. Hence $X_{w_0} = \frac{1}{6}\alpha\beta(\alpha+\beta)$. As a check of this we compute the Jacobian J of the fundamental invariant. Recall

$$\sigma_1 = -(e_2+e_3)(e_2+e_3) + e_2 e_3$$

and

$$\sigma_2 = -(e_2+e_3)e_2 e_3,$$

where we have eliminated $e_1 = -(e_2+e_3)$. Then:

$$J = 3(e_2^2 e_3 - e_3^2 e_2) + 2(e_2^3 - e_3^3) = d,$$

so also, $X_{w_0} = \frac{1}{6} J$. Now by (3.7) we can compute

$$\Delta_\alpha\left(\frac{d}{6}\right) = \tfrac{1}{6}(2d_\alpha) = \tfrac{1}{3}\beta(\alpha+\beta)$$

and

$$\Delta_\beta \Delta_\alpha\left(\frac{d}{6}\right) = \tfrac{1}{3}(\Delta_\beta d_\alpha)) = \tfrac{1}{3}(d_{\alpha,\beta} + d_{s_\beta(\alpha),\beta}) = \tfrac{1}{3}(\alpha+\beta+\alpha) = \tfrac{1}{3}(2\alpha+\beta)$$

so that:

$$X_{s_\alpha s_\beta} = \tfrac{1}{3}\beta(\alpha+\beta) \quad \text{and} \quad X_{s_\alpha} = \tfrac{1}{3}(2\alpha+\beta) = \omega_\alpha$$

as one easily checks.

Now since the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ we have

$$\alpha = 2\omega_\alpha - \omega_\beta$$
$$\beta = -\omega_\alpha + 2\omega_\beta$$

so for example

$$X_{s_\alpha s_\alpha} = \tfrac{1}{3}(-X_{s_\alpha} + 2X_{s_\alpha})(X_{s_\alpha} + X_{s_\beta})$$
$$= \tfrac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta}X_{s_\alpha} + 2X_{s_\beta}^2)$$

which will be confirmed further in the next section.

Remark. In the crystallographic case, it follows from the Weyl denominator formula (see [6, p. 185], [2, p. 17]) that

$$\frac{d}{|W|} \equiv \frac{\rho^N}{N!} \pmod{I_W}$$

where ρ is the sum of the fundamental weights. Hence one can attempt to compute the operators Δ_w on ρ^N.

It is possible to develop such formulae and we hope to treat them elsewhere. In particular, one might want to conjecture in the general case that $\rho^N \notin I_W$, maybe even for all ρ in the interior of the fundamental chamber.

4. Pieri formula

Recall that the algebra of operators $\mathbb{A}_W$ was generated by both the Δ_α's and the multiplication operators ω^*. Using the basis constructed in (2.9), if one composes such operators, say $\omega^* \circ \Delta_w$ or $\Delta_w \circ \omega^*$, it should be possible to express them linearly in terms of the operators Δ_g, $g \in W$. Of course, our eventual concern is with the algebra $\mathbb{A}_W$ and

$$\varepsilon \circ \omega^* \cdot \Delta_w = 0 .$$

So, if we compute the commutator $[\Delta_w, \omega^*]$ a quick application of ε will yield a formula for $\varepsilon \cdot \Delta_w \circ \omega^*$. Here we are following the strategy of Bernstein-Gelfand-Gelfand [2]. Essentially, this result is our Pieri formula disguised in its dual form.

In order for the techniques of section 1 and induction to be easily applicable, we work with the slightly modified operator $w^{-1} \Delta_w$ (recall $W \subset \mathbb{A}_W$). The main result is

THEOREM 4.1. *If* $w \in W, \omega \in V^*$, *then in* End $S(V)$,

$$[w^{-1}\Delta_w, \omega^*] = \sum_{w' \xrightarrow{\gamma} w} \left(w'^{-1}(\gamma)^v, \omega\right) w^{-1} \Delta_{w'} .$$

We will now fix a reduced decomposition $w = s_{\alpha_1}, \ldots, s_{\alpha_k}$ and write s_i for s_{α_i} and $w_i = s_{\alpha_n} \ldots s_{\alpha_i}$. First we have the following easy observation.

LEMMA 4.2. *Let* $\theta_i = s_k \ldots s_{i+1}(\alpha_i) = w_{i+1}(\alpha_i), 1 \leqslant i \leqslant k$. *Then*

(i) $w^{-1} \Delta_w = \Delta_{\theta_1} \Delta_{\theta_2} \cdots \Delta_{\theta_k}$

and

(ii) $s_{\theta_i}(w\hat{_i})^{-1} = w^{-1}$

Proof. Note by (2.2 ii, iv) $s_\alpha \Delta_\alpha = \Delta_\alpha$. Hence

$$w^{-1}\Delta_w = s_k \dots s_1 \Delta_{\alpha_1} \dots \Delta_{\alpha_k} = \Delta_{s_k \dots s_1(\alpha_1)} s_k \dots s_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k}$$
$$= \Delta_{\theta_1} w_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k}$$

and induction completes the argument. The second remark follows precisely as in (1.3).

Proof of (4.1). We compute

$$[w^{-1}\Delta_w, \omega^*] = [\Delta_{\theta_1} \circ \dots \circ \Delta_{\theta_k}, \omega^*]$$
$$= \sum_{j=1}^{k} \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \dots \Delta_{\theta_k} \ .$$

Let us call the j-th summand P_j. Firstly, observe that $[\Delta_{\theta_j}, \omega^*] = (\theta_j^v, \omega)\, s_{\theta_j}$ by (2.2 vii). If we substitute this into P_j and drag the reflection s_{θ_j} over to the left we get

$$P_j = \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_j^v, \omega) \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} s_{\theta_j} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_v^j, \omega)\, s_{\theta_j} \Delta_{s_{\theta_j}(\theta_1)} \dots \Delta_{s_{\theta_j}(\theta_{j-1})} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_j^v, \omega)\, s_{\theta_j} (w\hat{j})^{-1} \Delta_{w\hat{j}} \ .$$

To see this final identity we must argue, by (4.2), that $s_{\theta_j}(\theta_i) = \pm\, \theta_{i,\hat{j}}$ where $\theta_{i,\hat{j}} = s_k \dots \hat{s}_j \dots s_{i+1}(\alpha_i)$. (As in the above remark, $\theta_{i.\hat{j}}$ is the θ_i for $w\hat{j} = s_1 \dots \hat{s}_j \dots s_k$.) But, we can assume $i < j$, in which case

$$s_{\theta_j}(\theta_i) = s_k \dots s_{j+1} s_j s_{j+1} \dots s_k (s_k \dots s_j s_{j+1} \dots s_{i+1}(\alpha_i))$$
$$= s_k \dots \hat{s}_j \dots s_{i+1}(\alpha_i) = \theta_{i,\hat{j}} \ .$$

And now, by (4.2 ii)

$$P_j = (\theta_j^v, \omega)\, w^{-1} \Delta_{w\hat{j}}$$

and $s_{w\hat{j}(\theta j)}(w\hat{j}) = w$, so $w\hat{j} \xrightarrow{w\hat{j}(\theta j)} w$. Finally, (1.2) allows us to reindex by the immediate subwords of w

$$\sum_{j=1}^{k} P_j = \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma)^v, \omega)\, w^{-1} \Delta_{w'}$$

and the proof is complete.

COROLLARY 4.3.

$$\Delta_w \cdot \omega^* = w \cdot \omega^* \cdot w^{-1} \Delta_w + \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma)|, \omega(\Delta_{w'} .$$

Proof. Multiply (4.1) by w.

COROLLARY 4.4.

$$\varepsilon \cdot \Delta_w \cdot \omega^* = \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma^v), \omega) \varepsilon \cdot \Delta_{w'} .$$

Proof. The first term in the right-hand side of (4.3) is annihilated by ε. It is now easy to dualize the above and obtain

THEOREM 4.5 (Pieri formula). *If* $w \in W, \alpha \in \Sigma$, *then in* H_W

$$X_{s_\alpha} \cdot X_w = \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^v, w_\alpha) X_{w'} .$$

Proof. Choose A such that $\varepsilon \cdot \Delta_{w'}(A) = \delta_{ww'}$, for example $\sigma(X_w)$. Then, by (3.4 ii)

$$\begin{aligned}
X_{s_\alpha} \cdot X_w &= c(\omega_\alpha A) \\
&= \sum_{w' \in W} \varepsilon \Delta_{w'}(\omega_\alpha A) X_{w'} \\
&= \sum_{w' \in W} \varepsilon \cdot \Delta_{w'} \omega_\alpha^*(A) X_{w'} \\
&= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \omega_\alpha) \varepsilon \circ \Delta_g(A) \Big) X_{w'} \\
&= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \omega_\alpha) \delta_{gw} \Big) X_{w'} \\
&= \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^v, \omega_\alpha) X_{w'} .
\end{aligned}$$

Of course, it is also possible to rewrite this formula in the following equivalent form.

COROLLARY 4.6.

$$X_{s_\alpha} \cdot X_w = \sum_{\substack{\beta \in \Delta + \\ l(ws_\beta) = l(w) + 1}} (\beta^v, \omega_\alpha) X_{ws_\beta} .$$

Proof. It suffices to note $\sigma_\gamma w = w'$ if and only if $w \sigma_{w^{-1}(\gamma)} = w'$.

Example. Recall that in H_{Σ_3} we computed

$$X_{s_\alpha s_\beta} = \tfrac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2) .$$

By (4.6), one can compute

$$\begin{aligned} X_{s_\alpha}^2 &= X_{s_\beta s_\alpha} \\ X_{s_\beta} X_{s_\alpha} &= X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta} \\ X_{s_\beta}^2 &= X_{s_\alpha s_\beta} \end{aligned}$$

and this confirms our earlier computation.

5. H_W AS A W-MODULE AND PARABOLICS

If (W, S) is a Coxeter system and $\theta \subseteq S$ then (W_θ, θ) is also a Coxeter system [6, p. 20] and W_θ is called a *parabolic* subgroup of W. In addition, it is easy to see that a geometric realization (Δ, Σ) of (W, S) can be restricted to a geometric realization of (W_θ, θ). The collection $\{W_\theta\}_{\theta \subset S}$ of parabolic forms a lattice of $2^{|S|}$ distinct subgroups where, for example, $W_\theta \cap W_{\theta'} = W_{\theta \cap \theta'}$. We will eventually be concerned with the set of left cosets of W_θ in W. We define $W^\theta = \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$. The following basic result is well-known [6, p. 37 and p. 45].

THEOREM 5.1. *Every element w of W can be uniquely expressed as $w^\theta \cdot w_\theta$ with $w^\theta \in W^\theta$, $w_\theta \in W_\theta$ and furthermore $l(w) = l(w^\theta) + l(w_\theta)$.*

This immediately yields

COROLLARY 5.2. *W^θ is a complete set of left coset representations for W_θ in W and furthermore provides an element of the coset of minimal length.*

In this section we analyze the subalgebra $H_W^{W_\theta}$ of W_θ-invariants in H_W. The most straightforward approach is to compute exactly the action of W on H_W. This is easily done by exploiting the computation (4.1).

THEOREM 5.3. *The structure of H_W as a W-module is given by*

$$s_\alpha \cdot X_w = \begin{cases} X_w & \text{if } l(ws_\alpha) = l(w) + 1 \\ X_w - \displaystyle\sum_{ws_\alpha \xrightarrow{\gamma} w'} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'} & \text{if } l(ws_\alpha) = l(w) - 1. \end{cases}$$

Proof. As in (4.5), choose A such that $\varepsilon \Delta_g(A) = \delta_{gw}$. Then, since c is a W-map

$$\begin{aligned}
s_\alpha X_w &= c(s_\alpha A) = \sum_{w' \in W} \varepsilon \Delta_{w'}(s_\alpha A) X_{w'} \\
&= \sum_{w' \in W} \varepsilon \Delta_{w'}(1 - \alpha^* \Delta_\alpha)(A) X_{w'} \\
&= X_w - \sum_{w' \neq w} (\varepsilon \Delta_{w'} \alpha^*) \Delta_\alpha(A) X_{w'} \\
&= X_w - \sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \alpha) \varepsilon \Delta_{gs_\alpha}(A) X_{w'} \qquad \text{by (4.1)} \\
&= \sum_{\substack{g \xrightarrow{\gamma} w' \\ l(gs_\alpha) = l(g) + 1 \\ gs_\alpha = w}} (g^{-1}(\gamma)^v, \alpha) X_{w'} \\
&= X_w - \sum_{ws_\alpha \xrightarrow{\gamma} w'} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'}
\end{aligned}$$

Note, that the summation in the next to the last line is non-vacuous if and only if $l(ws_\alpha) = l(w) - 1$. This completes the proof.

COROLLARY 5.4. $X_w \in H_W^{W_\theta}$ *if* $w \in W^\theta$.

Proof. Immediate from (5.3) and the definition of W^θ.

The following elementary result shows that the X_w, $w \in W^\theta$, are actually an **R**-basis for $H_W^{W_\theta}$.

LEMMA. *If a finite group* G *acts on a real vector space* V *via the regular representation and* H *is a subgroup of* G, *then*

$$\dim_{\mathbf{R}} (V^H) = [G : H].$$

Proof. Let $\{e_g\}_{gG}$ be a basis for V, so that

$$g' \cdot e_g = e_{gg'}$$

Then if $\chi = \sum_{g \in G} \xi_g \in V^H$, we claim $\xi_g = \xi_{g'}$, if $g \equiv g' \pmod{H}$. Indeed, if $g = g' h$, $h \in H$, then

$$\begin{aligned}
\xi_{g'} &= \text{coefficient of } e_{g'}\ \chi \text{ in} \\
&= \text{coefficient of } e_{g'} \text{ in } h^{-1} \chi \cdot \\
&= \text{coefficient of } e_{g'h} \text{ in } \chi \\
&= \xi_g .
\end{aligned}$$

Hence, there are at most $[G : H]$ free parameters in determining $\chi \in V^H$ and clearly each choice gives an invariant. This finishes the proof.

COROLLARY 5.6. *dim* $(H_W^{W_\theta}) = [W : W_\theta] = |W^\theta|$ *and the* X_w, $w \in W^\theta$, *are an* **R**-*basis for* $H_W^{W_\theta}$.

Proof. Chevalley [8] has shown that S_W, hence H_W, is abstractly equivalent to the regular representation of W, as a W-module. Hence, (5.5) applies and the result follows.

It is now possible to "restrict" the Pieri formula (4.5) for H_W to $H_W^{W_\theta}$. We have

THEOREM 5.7. *If* $w, w' \in W^\theta$ *and in* H_W

$$X_w \cdot X_{w'} = \sum_{w'' \in W} c(w, w', w'') X_{w''}$$

then in $H_W^{W_\theta}$

$$X_w \cdot X_{w'} = \sum_{w'' \in W^\theta} c(w, w', w'') X_{w''}$$

Proof. One need only observe that the vector space map $r : H_W \to H_W^{W_\theta}$ given by

$$r(X_w) = \begin{cases} X_w & \text{if } w \in W^\theta \\ 0 & \text{otherwise} \end{cases}$$

is a retraction. Then, applying r to both sides of the first equation yields the second equation since the invariants form a subalgebra.

This result will be useful in the next section for computing inside the algebra of W_θ-invariants. Notice that an appropriate Giambelli formula for $H_W^{W_\theta}$ is not as easily obtained. This is because the Giambelli formula for H_W gives X_w as a polynomial in the X_{s_α}'s and not all of these are in the algebra $H_W^{W_\theta}$, so this is not quite the right thing.

6. APPLICATION: THE COMBINATORICS OF THE CLASSICAL PIERI FORMULA

In the last section we saw that given a pair (W, W_θ) of a Coxeter group and a parabolic subgroup, one could construct a formula to describe the multiplication of Schubert generators in the invariant subalgebra $H_W^{W_\theta}$. In this section, we examine the particular case $(\Sigma_{n+k}, \Sigma_k \times \Sigma_n)$ where Σ_m denotes the symmetric group on m letters. Indeed, Σ_{n+k} is the Weyl group of the root system of type A_{n+k-1}, which we recall briefly here. Let $V' = \mathbf{R}^{n+k}$

equipped with the usual inner product and let $e_1, ..., e_{n+k}$ denote the standard basis. Σ_{n+k} acts by permuting these basic elements. This action is effective on the $(n+k-1)$-dimensional subspace

$$V = \left\{ \sum_{i=1}^{n+k} \lambda_i e_i : \sum_{i=1}^{n+k} \lambda_i = 0 \right\}$$

and it is easy to see $\Delta = \{e_i - e_j\}_{j \neq i}$ can be chosen as the corresponding root system. In addition, the simple roots $\Sigma = \{e_i - e_{i+1}\}_{1 \leq i \leq n+k-1}$ and the positive roots $\Delta^+ = \{e_i - e_j\}_{i<j}$, induce the usual transpositions of the basis vectors.

The main result of this section is the identification of the Pieri formula for $H_{\Sigma_{n+k}}^{\Sigma_k \times \Sigma_n}$ with the classical Pieri formula (see [7, 16]).

We begin with a rapid review of Chern's Schubert calculus for the cohomology of a complex grassmannian [7]. Let $G_k(\mathbf{C}^{n+k})$ denote the space of k-dimensional complex subspaces in $\mathbf{C}^{n+k}$. This is a compact, smooth manifold of dimension $2nk$. Ehresmann [14] described a cell-decomposition for $G_k(\mathbf{C}^{n+k})$ (along with other algebraic homogeneous spaces) whose cells are identified by certain Schubert symbols $(d_1, ..., d_k)$, where

$$1 \leqslant d_1 < ... < d_k \leqslant n + k\,.$$

Each symbol yields a cohomology class $\langle d_1 ... d_k \rangle$ of dimension

$$2 \sum_{i=1}^{k} (d_i - 1) = \left(\sum_{i=1}^{k} d_i \right) - \frac{k(k+1)}{2}$$

Geometrically, $\langle d_1, ..., d_k \rangle$ is the cocycle dual to the cell

$$[d_1, ..., d_k] = \{X \in G_k(\mathbf{C}^{n+k}) : \dim (X \cap \mathbf{R}^{d_i}) \geqslant i\}\ .$$

It is easy to see the d_i's describe the "jump-points" in the sequence

$$0 \leqslant \dim (X \cap \mathbf{C}^1) \leqslant \dim (X \cap \mathbf{C}^2) \leqslant ... \leqslant \dim (X \cap \mathbf{C}^{n+k-1}) \leqslant n + k$$

where $0 \subseteq \mathbf{C}^1 \subseteq \mathbf{C}^2 \subseteq ... \subseteq \mathbf{C}^{n+k}$ is the standard flag determined by the coordinate axes.

On the other hand, $G_k(\mathbf{C}^{n+k})$ can also be profitably thought of as the homogeneous space G/P where G is the complex Lie group $GL_{n+k}(\mathbf{C})$ and P is the maximal parabolic subgroup of the form

$$\left(\begin{array}{c|c} GL_k(\mathbf{C}) & * \\ \hline 0 & GL_n(\mathbf{C}) \end{array} \right)$$

If K denotes the maximal compact subgroup U_{n+k} of G then we also have the identification $G_k(\mathbf{C}^{n+k}) = K/(U_k \times U_n)$.

More generally, one can consider a complex semisimple Lie group G and a parabolic P_θ in G corresponding to a subset θ of the simple roots Σ. The homogeneous space G/P_θ has been studied by various authors and we will assume known that

$$H^*(G/P_\theta; \mathbf{R}) \cong H^*(G/B; \mathbf{R})^{W_\theta} \cong (S_W)^{W_\theta} .$$

This will be the basic topological input [2].

Now we fix G to be the Lie group of type A_{n+k-1} and $\theta = \Sigma - \{\alpha_k\}$ (where we write $\alpha_j = e_j - e_{j+1}$ and $s_j = s_{\alpha_j}$) so that $G_k(\mathbf{C}^{n+k}) \cong G/P_\theta$. We begin with some easy length computations.

LEMMA 6.1. *If* $w \in W$, *then*

$$l(ws_{ij}) - l(w) = p_{ij}(2\,|I_{i,j}| + 1)$$

where

$$p_{ij} = \begin{cases} +1 & w(i) < w(j) \\ -1 & w(i) > w(j) \end{cases}$$

and

$$I_{i,j} = \{ i < z < j : w(z) \text{ is between } w(i) \text{ and } w(j) \} .$$

In particular, $l(ws_{ij}) = l(w) + 1$ *if and only if* (i) $w(i) < w(j)$ *and* (ii) *there are no intermediate* w*-values, i.e.* $I_{i,j} = \phi$ *(we often abbreviate this pair of conditions by* $w(i) << w(j)$.

Proof. Recall the length function on $\sum_{n+k}$ is given by $l(w) = \sum_{j=1}^{n-1} e_j(w)$, where $e_j = |\{i > j : w(i) < w(j)\}|$, the number of inversions of w. Hence

$$l(ws_{ij}) - l(w) = (e_i' - e_i) + (e_j' - e_j) + \sum_{i<z<j} (e_z' - e_z)$$

where $e_l = e_l(w)$ and $e_l = e_l(ws_{ij})$. Certainly, right multiplication by s_{ij} does not affect the values of e_l below i or above j. Also

$$e_i' = e_j + |\{ i \leqslant z < j : w(z) < w(j) \}| = e_j + e$$
$$e_j' = e_i - |\{ i < z \leqslant j : w(z) < w(i) \}| = e_j - \bar{e}$$

so we get

$$(e_i' - e_i) + (e_j' - e_j) = (e_j + e - e_i) + (e_i - \bar{e} - e_j) = e - \bar{e} = p_{i,j}(|I_{i,j}| + 1)$$

It is easy to see

$$e_z' - e_z = \begin{cases} p_{i,j} & \text{if } z \in I_{i,j} \\ 0 & \text{otherwise} \end{cases}$$

putting this all together we get the result. The second assertion follows immediately.

We can now write down (4.6) for H_W, $W = \sum_{n+k}$

PROPOSITION 6.2. *If* $w \in \sum_{n+k}$, $1 \leqslant i \leqslant n+k-1$, *then in* $H_{\Sigma_{n+k}}$

$$X_{s_i} \cdot X_w = \sum_{(b,t)} X_{ws_{bt}}$$

where (b, t) *satisfies* $b \leqslant i < t$ *and* $w(b) << w(t)$.

Proof. By (4.6), $X_{ws_{bt}}$ appears with coefficient $((e_b - e_t)^v, \omega_i)$ if and only if $l(ws_{bt}) = l(w) + 1$. This is equivalent to the last condition by (6.1). Finally $(e_b - e_t)^v = e_b - e_t = \alpha_b + \ldots + \alpha_{t-1}$, so that first condition is also needed and the coefficient is correct.

Remark. The Poincaré dual of this formula appears in [18, p. 265].

We now identify the set of left coset representatives W^θ. If $1 \leqslant d_1 < \ldots < d_k \leqslant n+k$ and $d_1' < \ldots < d_n'$ is an ordered enumeration of the complement then we define $(d_1, \ldots, d_k) \in \sum_{n+k}$, by

$$(d_1, \ldots, d_k)(i) = \begin{cases} d_i & 1 \leqslant i \leqslant k \\ d_{i-k}' & k+1 \leqslant i \leqslant k+n \end{cases}.$$

(We also write $X(d_1, \ldots, d_k)$ when it is convenient.)

LEMMA 6.3.

$$W^\theta = \{(d_1, \ldots, d_k) : 1 \leqslant d_1 < \ldots < d_k \leqslant n+k\}$$

and $\quad l(d_1, \ldots, d_k) = \sum_{j=1} (d_j - j).$

Proof. Clearly $l((d_1, \ldots, d_k)\, s_i) = l(d_1, \ldots, d_k) + 1$, for all $i \neq k$ by (6.1), for example. Since $|W^\theta| = |W| / |W_\theta| = \binom{n+k}{k}$ the first assertion follows. For the second, we need only observe

$$e_j(d_1, \ldots, d_k) = \begin{cases} d_j - j & \text{if } j \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

This lemma indicates how the Schubert notation arises from a group-theoretic point of view. That this notation is consistent with the geometry is a theorem of Demazure [12].

A Pieri formula should compute the product of X_w (a linear generator, by (5.6)) and an algebraic generator. Since the map

$$S(V)^{W_\theta} \xrightarrow{c} (H_W)^{W_\theta}$$

is onto we can find algebraic generators by computing the images of W_θ-invariants. In general, W_θ is a (reducible) Coxeter group, so we have the fundamental invariants [20]. In our case, we have simply

$$S(V)^{W_\theta} = \mathbf{Z}[\tau_1, \ldots, \tau_k, \sigma_1, \ldots, \sigma_n]$$

where $\tau_i = s_i(e_1, \ldots, e_k)$, $1 \leqslant i \leqslant k$, and $\sigma_j = s_j(e_{k+1}, \ldots, e_{k+n})$, $1 \leqslant j \leqslant n$ and s_j denotes the j^{th} elementary symmetric function in an appropriate number of variables. One knows $c(\sigma_j)$ suffices to generate $H_W^{W_\theta}$. So we compute

LEMMA 6.4.

$$c(\sigma_j) = (-1)^j X_{s_{k+j-1}, \ldots, s_k} = (-1)^j X(1, 2, \ldots, k-1, k+j) \quad .$$

Proof. By section 2

$$c(\sigma_j) = \sum_{l(w) = j} \Delta_w(\omega_j) X_w$$

If we write Δ_t for Δ_{α_t}, then clearly $\Delta_t(\sigma_j) = 0$, if $t \neq k$ and

$$\begin{aligned} \Delta_k(\sigma_j) &= \frac{s_j(e_{k+1}, \ldots, e_{k+n}) - s_j(e_k, \ldots, e_{k+n})}{e_k - e_{k+1}} \\ &= \frac{(e_{k+1} - e_k)\, s_{j-1}(e_{k+2}, \ldots, e_{k+n})}{e_k - e_{k+1}} \\ &= (-1)\, s_{j-1}(e_{k+2}, \ldots, e_{k+n}) \end{aligned}$$

We can continue by induction and get $\Delta_{k+j-1} \ldots \Delta_k(\sigma_j) = (-1)^j$, while any other sequence of simple roots yields zero.

We now proceed to a computation of

$$X(1, 2, \ldots, k-1, k+j)\, X(d_1, \ldots, d_k) .$$

The case $j = 1$ is easy.

PROPOSITION 6.5.

$$X(1, 2, \ldots, k-1, k+1)\, X(d_1, \ldots, d_k) = \sum_{d_i + 1 < d_{i+1}} X(d_1, \ldots, d_i + 1, \ldots, d_k) \; .$$

Proof. Since $(1, 2, \ldots, k-1, k+1) = s_k$, we apply the case $i = k$ of (6.2) and observe $w(b) \ll w(t)$ if and only if $w(t) = w(b) + 1$.

We now observe

LEMMA 6.6.

$$c(\sigma_j) = s_j(X_{s_{k+1}} - X_{s_k}, X_{s_{k+2}} - X_{s_{k+1}}, \ldots, -X_{s_{n+k-1}})$$

Proof. By the tables of [6], the i-th fundamental weight is

$$\omega_i = e_1 + \ldots + e_i - \left(\frac{i}{n+k}\right) \sigma_1(e_1, \ldots, e_{n+k}) \; .$$

Hence $\omega_i \equiv e_1 + \ldots + e_i \pmod{I_w}$ and we get

$$\begin{aligned} c(\sigma_j) &= c\big(s_j(e_{k+1}, \ldots, e_{k+n})\big) \\ &= c\big(s_j(\omega_{k+1} - \omega_k, \ldots, -\omega_{n+k-1})\big) \\ &= s_j(X_{s_{k+1}} - X_{s_k}, \ldots, -X_{s_{n+k-1}}) \end{aligned}$$

since c kills I_W and (3.4 ii).

This suggests the following computation.

LEMMA 6.7. *For all* $i, k + 1 \leqslant i \leqslant k + n, w \in W$; *in* H_W

$$(X_{s_i} - X_{s_{i+1}}) X_w = \sum_{\substack{i < t \\ w(i) << w(t)}} X_{ws_{it}} - \sum_{\substack{k < b < i \\ w(b) << w(i)}} X_{ws_{bi}} - \sum_{\substack{b \leqq k \\ w(b) << w(i)}} X_{ws_{bi}}$$

Proof. Computing with (6.2), we get

$$X_{s_i} X_w = \sum_{\substack{b \leqq i-1 \\ i < t \\ w(b) << w(t)}} X_{ws_{bt}} + \sum_{\substack{i < t \\ w(i) << w(t)}} X_{ws_{it}}$$

and

$$X_{s_{i-1}} X_w = \sum_{\substack{b \leq i-1 \\ i < t \\ w(b) << w(t)}} X_{ws_{bt}} + \sum_{\substack{b < i \\ w(b) << w(i)}} X_{ws_{bi}}$$

Upon subtracting and breaking up the second term the desired expression follows.

THEOREM 6.8. *In*

$$s_j(X_{s_{k+1}} - X_{s_k}, \ldots, -X_{s_{n+k-1}}) X(d_1, \ldots, d_k) = (-1)^j \Sigma X(e_1, \ldots, e_k)$$

where the summation ranges over $(e_1, \ldots, e_k)$ *satisfying* $d_i \leqslant e_i \leqslant d_{i+1}$ *and* $\sum_{i=1}^{k} e_i = j + \sum_{i=1}^{k} d_i$.

Proof. Of course

$$s_j = \sum_{k+1 \leq t_1 < \ldots < t_j \leq k+n} (X_{s_{t_j}} - X_{s_{t_j-1}}) \ldots (X_{s_{t_1}} - X_{s_{t_1-1}})$$

where we set $X_{s_{k+n}} = 0$. It is not difficult to check that the third term of (6.7) alone yields the right-hand side. Hence it remains to show that the contributions arising whenever either of the first two terms of (2.7) are involved cancel in the final summation. To do this it suffices to show that the resulting subscripts in W do not lie in W^θ. (Then they must have coefficient zero since $H_{W^\theta}^{W}$ is a subalgebra of H_w.)

Now the first two terms of (6.7) always give a transposition above $k+1$ and it must be elementary one by (6.1), say s_i, $i \geqslant k$. Such a transposition will send an element of W^θ out of W^θ. We claim no further transposition s_{bt}, with either $b \geqslant i$ or $t \geqslant i$, will put the subscript back in W^θ. Both cases are easy to check and the proof is complete. Finally, by a substitution from (6.4) we get

COROLLARY 6.9. (Pieri formula). *In* $H_{\Sigma_{n+k}}^{\Sigma_k \times \Sigma_n} = H^*(G_k(\mathbf{C}^{n+k}))$

$$X(1, 2, \ldots, k-1, k+j)\, X(d_1, \ldots, d_k) = \Sigma X(e_1, \ldots, e_k)$$

where the summation is as in (6.8).

Remarks 1. The above formal arguments are equally valid for the Chow ring of the grassmann variety over an arbitrary algebraically closed field.

2. One can hope to mimick the above strategy for the homogeneous space SO_{2n+1}/U_n. The group G is of type B_n and the maximal parabolic is determined by the "right-end" root. It is not difficult to write out the Pieri formula for the flag manifold of type B_n (see the author's "Pieri formulae for classical groups", preprint). In addition, W^θ, for this case, can be identified with the power set of $\{1, 2, ..., n\}$ and one can compute $c(\sigma_j) = 2X_{\{j\}}$. (The 2 occurs because c is not onto.) Still the problem is complicated by multiplicities. We hope to return to this elsewhere.

REFERENCES

[1] Artin, E. *Galois Theory*. Notre Dame Math. Lect., Notre Dame, 1958.

[2] Bernstein, I. N., I. M. Gelfand and S. I. Gelfand. Schubert cells and the cohomology of the spaces *G/P. Russian Math. Surveys 28* (1973), 1-26.

[3] Borel, A. Sur la cohomologie des espaces fiibres principaux et des espaces homogenes de groupes de Lie compacts. *Ann. Math. 57* (1953), 115-207.

[4] Borel A. and F. Hirzebruch. Characteristic classes and homogeneous spaces. *Amer. J. Math. 81* (1959), 315-382.

[5] Borel, A. Sous groupes commutatifs et torsion des groupes de Lie compacts connexes. *Tohoku Math. J. 13* (1961), 216-240.

[6] Bourbaki, N. *Groupes et algèbres de Lie. Chapitres IV, V et VI.* Paris, Hermann, 1968.

[7] Chern, S. On the multiplication in the characteristic ring of a sphere bundle. *Ann. Math. 49* (1948), 362-372.

[8] Chevalley, C. Invariants of finite groups generated by reflections. *Amer. J. Math. 77* (1955), 778-782.

[9] Coxeter, H. S. M. Discrete groups generated by reflections. *Ann. Math. 35* (1934), 588-621.

[10] —— *Regular Polytopes.* MacMillan, New York, 1963.

[11] Demazure, M. Invariants symétriques entiers des groupes de Weyl et torsion. *Inv. Math. 21* (1973), 287-301.

[12] —— Désingularisation des variétés de Schubert généralisées. *Ann. Scient. Ec. Norm. Sup., 4e Serie, t. 7* (1974), 53-88.

[13] Deodhar, V. Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Môbius function. *Inv. Math. 39* (1977), 187-198.

[14] Ehresmann, C. Sur la topologie de certains espaces homogènes. *Ann. Math. 35* (1934), 396-443.

[15] Garland, H. and M. Raghunathan. A Bruhat decomposition for the loop space of a compact group: a new approach to results of Bott. *Proc. Nat. Acad. Sci., U.S.A. 72* (1975), 4716-4717

[16] Kleiman, S. Problem 15. *Rigorous foundations of Schubert's enumerative calculus.* Proc. Symp. Pure Math., vol. 28, Amer. Math. Soc., Providence, R.I., 1976.

[17] Kostant, B. Lie algebra cohomology and generalized Schubert cells. *Ann. Math. 77* (1963), 72-144.

[18] Monk, D. The geometry of flag manifolds. *Proc. London Math. Soc. 9* (1959), 253-286.

[19] SOLOMON, L. Invariants of finite reflection groups. *Nagoya Math. J. 22* (1963), 57-64.
[20] SPRINGER, T. A. *Invariant theory.* Lecture Notes in Mathematics, 585, Springer Verlag, New York, 1977.
[21] STEINBERG, R. Differential equations invariant under finite reflection groups. *Trans. Amer. Math. Soc. 112* (1964), 392-400.
[22] STOLL, W. *Invariant forms on grassmann manifolds.* Annals of Math. Studies, Princeton University Press, Princeton, N.J. 1977.

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