# Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **27 (1981)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

# SCHUBERT CALCULUS OF A COXETER GROUP

by Howard L. HILLER [1])

## INTRODUCTION

Let $(W, S)$ be a finite Coxeter system, in the sense of Bourbaki [6]. It is well-known that $W$ can be realized as the Weyl group of a (possibly non-crystallographic) root system $\Delta$ in a real Euclidean space $V$ of dimension $n = |S|$. This space possesses a basis $\Sigma$ of simple roots such that the reflections $s_\alpha$ through the hyperplane perpendicular to $\alpha \in \Sigma$ precisely yield the generating set $S$. In this fashion, $W$ admits a natural representation on $V$, so we can make it act on the polynomial algebra $S(V)$ on $V$ by $w.f(x) = f(w^{-1}x)$. The invariant subalgebra splits up into its homogenous components $S(V)^W = \bigoplus_{j=0}^{\infty} S_j(V)^W$ and the positive components generate a graded homogenous ideal $I_W$. We can form the quotient algebra $S_W = S(V)/I_W$ which we refer to as the *coinvariant algebra* of $W$. Of course, Chevalley's theorem [8] tells us that $S(V)^W$ has $n$ algebraically independent generators whose degrees $d_1, \ldots, d_n$ (the fundamental degrees) are useful in describing the gross structure of $S_W$. In particular, one can compute the Poincaré series of $S_W = \bigoplus_j S_{W,j}$

$$PS(S_W, t) = \sum_{j=0}^{\infty} \dim_{\mathbf{R}} (S_{W,j})\, t^j = \prod_{i=1}^{n} (1+t+\ldots+t^{d_i-1})$$

so that the real dimension is $PS(S_W, 1) = \prod_{i=1}^{n} d_i = |W|$ and $S_{W,j} = 0$, $j > \deg(PS(S_{W,t})) = \sum_{i=1}^{n} (d_i - 1)$. Note that the last sum is also equal to the number $N$ of reflections in $W$, for example, by a formula of Solomon [19].

We are interested here in a finer analysis of the algebraic and $W$-module structure of $S_W$. Following Demazure [11], we describe a sort of algebraic

[1]) Partially supported by an American Mathematical Society Post-Doctoral Research Fellowship.

Bruhat decomposition for $S_W$ relative to a root system for $W$ (section 2). By this we mean an algebra $H_W$ is constructed with a basis $\{X_w\}_{w \in W}$ and a map $c: S(V) \to H_W$, that induces an isomorphism $S_W \approx H_W$. The basis depends on the relative lengths of the simple roots. In the case where $W$ is a Weyl group, and the lengths of the roots are chosen to make the Cartan matrix integral, the element $X_w$ corresponds to the cocyle dual to the Ehresmann-Bruhat cell decomposition of a certain flag manifold $G/B = K/T$. Hence, for example, the Coxeter group $\Sigma_n \int \mathbf{Z}_2$ admits the two different Schubert calculi of type $B_n$ and $C_n$. In addition, the map $c$ above corresponds to taking the first Chern class of the line bundle associated to a character of $T$ (where $V$ is thought of as the character group $X(T)$ on the maximal torus). Our first task is to describe a section for the map $c$ (section 3). We think of this as a Giambelli formula for $S_W$. This leads us to introduce a notion of fundamental weights for a Coxeter system, which turns out to yield the 1-dimensional generators $X_{s_\alpha}$, $\alpha \in \Sigma$. This allows us to view an arbitrary $X_w$ as a polynomial in the $X_{s_\alpha}$'s.

In section 4, we look closer at the multiplicative structure of $H_W$. By our Giambelli formula, it suffices to understand multiplication of $X_w$ by a fundamental weight. Here we exploit a commutator computation of Bernstein, Gelfand and Gelfand [2] to get such a Pieri formula.

It is possible to relativize the above results. In section 5 we recall the basic facts about the lattice of parabolic subgroups $\{W_\theta\}_{\theta \subset S}$ of the Coxeter group $W$. Of course, $(W_\theta, \theta)$ is a Coxeter system itself. We consider the invariant algebra $H_W^{W_\theta}$ and show that it is generated by $\{X_w\}_{w \in W^\theta}$ where $W^\theta$ is a familiar set of coset representatives for $W_\theta$ in $W$.

Finally, using the results of section 5 and the parabolic $\Sigma_k \times \Sigma_n \subset \Sigma_{n+k} = W(A_{n+k-1})$ ge give an algebraic derivation of the classical Pieri formula of the Schubert calculus of a complex grassmannian.

Section 1 is a brief review of facts about Coxeter groups we will require in the sequel.

We note in passing that the sort of results described here have already been analyzed from a variety of viewpoints—the Chow ring [12], Lie algebra cohomology [17], and De Rham cohomology [22], to mention a few. The advantage of our method, inspired by [2] and [11] is that once one has identified the algebra in question as the coinvariant algebra $S_W$, all of the Schubert machinery follows in a purely formal fashion.

It is hoped that an extension of this circle of ideas to affine Weyl groups will shed some light on the Bott decomposition of the loop space of a Lie group [15].