THE RIEMANN-ROCH THEOREM FOR COMPACT RIEMANN SURFACES

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§ 1. Introduction

The aim of this article is to present a sheaf-theoretic proof of the Riemann-Roch theorem (including Serre duality) for vector bundles on compact Riemann surfaces. The basic assumption will be the finite dimensionality of cohomology vector spaces; no further potential theory will be used. Thus the proof will work (with trivial modifications) in the algebraic case also (over an algebraically closed field of any characteristic). The possibly new contribution of the article is a simple direct proof of the fact that the degree of the canonical divisor is $2g - 2$, where $g = \dim H^1(X, \mathcal{O})$.

We now give an outline of the contents. The rather long Section 2 gives the necessary definitions and sheaf-theoretic results, and the consequences of the finite dimensionality theorem which are needed later. Section 3 gives the preliminary form of the Riemann-Roch theorem. The identity $\deg K = 2g - 2$ is proved in Section 4. Serre duality and the final form of the Riemann-Roch theorem are proved in Section 5.

Our exposition borrows freely from those of Serre [5] and Mumford [4]. We should also mention the proof of the Riemann-Roch theorem given in Grauert-Remmert [1] (Ch. VII).

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§ 2. Line bundles and vector bundles. Sheaf theoretic preliminaries

In all that follows, X will denote a compact Riemann surface, i.e. a connected compact complex manifold of complex dimension 1; $\mathcal{O} = \mathcal{O}_X$ will denote its structure sheaf, i.e. the sheaf of germs of holomorphic functions on X.

For any sheaf $\mathscr{F}$ on X, and any $P \in X$, $\mathscr{F}_P$ denotes the stalk of $\mathscr{F}$ at P; for $U \subset X$ open, $\mathscr{F}(U)$ denotes the set of sections of $\mathscr{F}$ over U.

(2.1) *Definition.* A *vector bundle of rank* r on X is an $\mathcal{O}_X$-Module (i.e. a sheaf of $\mathcal{O}_X$-modules) which is locally $\mathcal{O}_X$-isomorphic to $\mathcal{O}_X + \ldots + \mathcal{O}_X$ (r times). A *line bundle* is a vector bundle of rank one.

(2.2) *Example—Definition.* A divisor $D = \sum_{P \in X} n(P)P$ on X is just an element of the free abelian group Div (X) on the set X. We write $D \geqslant 0$ if $n(P) \geqslant 0$ for all $P \in X$, and $D \geqslant D'$ for another $D' \in \text{Div}(X)$ if $D - D' \geqslant 0$. For any such $D \in \text{Div}(X)$, we define a line bundle $\mathcal{O}(D)$ as follows. Let $\mathcal{M} = \mathcal{M}_X$ be the sheaf of germs of meromorphic functions on X. Then, for any $U \subset X$ open, $\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) : \text{ord}_P f \geqslant -n(P) \text{ for all } P \in U\}$. Then, for any $P \in X$, it is clear that $t_P^{-n(P)}$ is a local generator for $\mathcal{O}(D)$ near P, where t_P is a uniformising parameter at P; thus $\mathcal{O}(D)$ is indeed a line bundle, which is an $\mathcal{O}$-submodule of $\mathcal{M}$. It is clear that $D \geqslant 0$ iff $\mathcal{O}(D) \supset \mathcal{O}$.

(2.3) *Example.* The *canonical line bundle* $K_X = K$ on X is just the sheaf of holomorphic 1-forms on X. Thus, if (U, z) is a coordinate chart on X, then $K(U)$ is the set of differential 1-forms $f\,dz$ on U with $f \in \mathcal{O}(U)$, so that K is clearly a line bundle.

(2.4) *Example—Proposition.* Let $f : X \to Y$ be a nonconstant holomorphic map of compact Riemann surfaces, and $\mathscr{V}$ a vector bundle on X. Then the direct image sheaf $f_0(\mathscr{V})$ of $\mathscr{V}$ by f is a vector bundle on Y.

Proof: Recall that, for any $U \subset Y$ open, $f_0(\mathscr{V})(U) = \mathscr{V}(f^{-1}(U))$ and that $g \in \mathcal{O}_Y(U)$ acts as multiplication by $g \circ f$. Now note that f is a *proper* map, and that $f^{-1}(Q)$ is a finite set for each $Q \in Y$. Also, for any $P \in X$, there exist uniformising parameters z and w at P and $f(P)$ respectively such that $w \circ f = z^n$ for some integer $n \geqslant 1$. Thus it is easily seen that it suffices to prove the following: for the map $f : z \to z^n$ of the unit disc U in $\mathbf{C}$ onto another copy W of it, $f_0(\mathcal{O}_U)$ is a free $\mathcal{O}_W$-Module (or rank n). But this is clear; in fact the functions $1, z, \ldots, z^{n-1}$, regarded as sections of $f_0(\mathcal{O}_V)$ over W, generate it over $\mathcal{O}_W$ and are independent everywhere on W.

(2.5) *Definition.* A *meromorphic section* σ of a vector bundle $\mathscr{V}$ on X is a holomorphic section σ of $\mathscr{V}$ over the complement of some finite set $S \subset X$ such that, for each $P \in S$, there exists a connected neighbourhood

U of P and an $f \not\equiv 0$ in $\mathcal{O}(U)$ so that $f\sigma$ extends to a holomorphic section of V over U.

(2.6) *Remark.* A meromorphic section of $\mathcal{O}_X$ is just a meromorphic function on X. It is clear that the set of meromorphic sections of a vector bundle $\mathscr{V}$ is a vector space over the field of meromorphic functions on X, of dimension $\leqslant 1$ if rank $\mathscr{V} = 1$.

(2.7) *Definition.* The *divisor* $\operatorname{div} \sigma$ *of a meromorphic section* $\sigma \not\equiv 0$ of a vector bundle $\mathscr{V}$ on X is $\Sigma\, n(P)\, P$, where, for each $P \in X$, $n(P)$ is the integer characterised by $t_P^{-n(P)} \sigma \in \mathscr{V}_P - \mathfrak{m}_P \mathscr{V}_P$; here t_P is a uniformising parameter at P, and $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_P$; $n(P)$ is the *order* of σ at P.

We shall now deduce from the finiteness theorem that every vector bundle has plenty of meromorphic sections. We first state the finiteness theorem explicitly:

(2.8) *Finiteness Theorem. For every vector bundle* $\mathscr{V}$ *on* X, $H^0(X, \mathscr{V})$ *and* $H^1(X, \mathscr{V})$ *are finite-dimensional vector spaces over* $\mathbf{C}$;

$$H^2(X, \mathscr{V}) = 0\,.$$

(2.9) *Remark.* The finite dimensionality of H^0 and H^1 can be deduced from Montel's theorem and the fact that a locally compact Hilbert space is finite dimensional; see e.g. Gunning [2], p. 59 or [1], Ch. VI. The vanishing of H^i for $i \geqslant 2$ follows from the Dolbeault resolution for $\mathscr{V}$, see e.g. Gunning-Rossi [3], pp. 184; another proof will be indicated in (2.17).

(2.10) PROPOSITION. *Every vector bundle* $\mathscr{V}$ *on* X *admits (infinitely many) meromorphic sections.*

Proof: Pick any $P \in X$, and let (U, z) be a coordinate system centred at P (i.e. $z(P) = 0$). Let $\mathfrak{U}$ be the covering $\{U, X - P\}$ of X. We may assume that there is an $\mathcal{O}_U$-isomorphism $\varphi : \mathcal{O}_U^r \to \mathscr{V} \,|\, U$ ($r = \operatorname{rank} \mathscr{V}$). Then the set of r-tuples of polynomials in $1/z$ can be regarded (via φ) as an infinite dimensional subspace W of $Z^1(\mathfrak{U}, \mathscr{V})$. Now $H^1(\mathfrak{U}, \mathscr{V}) \hookrightarrow H^1(X, \mathscr{V})$ is finite dimensional, hence the kernel W' of the natural map $W \hookrightarrow Z^1(\mathfrak{U}, \mathscr{V}) \to H^1(\mathfrak{U}, \mathscr{V})$ is infinite dimensional. It is clear that different elements of W' lead to different meromorphic sections of $\mathscr{V}$,

q.e.d.

(2.11) COROLLARY. *Every vector bundle* $\mathscr{V}$ *on* X *is an extension of line bundles of the form* $\mathcal{O}(D)$, $D \in \operatorname{Div} X$ *(i.e. there exists an exact sequence*

$0 \to \mathcal{O}(D) \to \mathcal{V} \to \mathcal{V}' \to 0$ *with* $\mathcal{V}'$ *a vector bundle); every line bundle* $\mathcal{L}$ *is isomorphic to* $\mathcal{O}$ (div σ) *for any meromorphic section* σ *of* $\mathcal{L}$.

Proof: Choose any meromorphic section $\sigma\,(\not\equiv 0)$ of $\mathcal{V}$, and let $D = \operatorname{div} \sigma$. Then multiplication by σ makes $\mathcal{O}(D)$ an $\mathcal{O}_X$-submodule of $\mathcal{V}$, and it is clear that $\mathcal{V} \,|\, \mathcal{O}(D)$ is again a vector bundle, q.e.d.

(2.12) COROLLARY. *Let* $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ *be the Riemann sphere, and* $\mathcal{L}$ *a line bundle on* $\mathbf{P}^1$. *Then* $\mathcal{L} \,|\, \mathbf{C}$ *is trivial, i.e.* $\mathcal{L} \,|\, \mathbf{C} \approx \mathcal{O}_{\mathbf{C}}$.

Proof: Let σ be a meromorphic section of $\mathcal{L}$ over $\mathbf{P}^1$, and $\operatorname{div} \sigma = \Sigma\, n(P)\, P$. If z denotes the coordinate function on $\mathbf{C}$, then clearly $\sigma' = \prod_{P \in \mathbf{C}} (z - z(P))^{-n(P)}.\ \sigma$ is a nowhere-vanishing section of $\mathcal{L}$ over $\mathbf{C}$, q.e.d.

In order to see when, for $D, D' \in \operatorname{Div}(X)$, $\mathcal{O}(D)$ and $\mathcal{O}(D')$ are isomorphic as line bundles, we begin with a definition:

(2.13) *Definition.* Let $D, D' \in \operatorname{Div} X$. Then D is *equivalent* to D' (notation: $D \sim D'$) if there exists a meromorphic function $f \not\equiv 0$ such that $D' = D + \operatorname{div} f$.

Remark. An f as in (2.13), if it exists, is clearly unique upto a nonzero constant factor (X compact!)

(2.14) PROPOSITION. *Let* $D, D' \in \operatorname{Div}(X)$. *Then* $D \sim D'$ *iff* $\mathcal{O}(D)$ *and* $\mathcal{O}(D')$ *are isomorphic.*

Proof: Note that any $\mathcal{O}_X$-linear map $\mathcal{M}_X \to \mathcal{M}_X$ is defined by multiplication by a unique meromorphic function, and that any $\mathcal{O}_X$-linear map $\mathcal{O}(D) \to \mathcal{O}(D')$ extends naturally to one of $\mathcal{M}_X$ into itself. Now multiplication by the meromorphic function f maps $\mathcal{O}(D)$ into $\mathcal{O}(D')$ iff $-D + \operatorname{div} f \geqslant -D'$, i.e. $D' + \operatorname{div} f \geqslant D$, so the proposition follows.

(2.15) *Remark.* The map $D \to \mathcal{O}(D)$ thus sets up a bijection between the set of equivalence classes of divisors and the set Pic X of isomorphism classes of line bundles on X. Now the group structure on Div X clearly induces one on the set of divisor classes, hence by the above map on Pic X. It is easy to see that the induced group operation on Pic X corresponds to the tensor product (over $\mathcal{O}_X$) of line bundles: multiplication of functions induces a canonical map $\mathcal{O}(D) \otimes \mathcal{O}(D') \to \mathcal{O}(D + D')$ which is clearly an isomorphism. It is also easy to verify that the inverse of the line bundle $\mathcal{L}$ is represented by $\operatorname{Hom}_{\mathcal{O}X}(\mathcal{L}, \mathcal{O}_X)$ (cf. the proof of (2.14)).

Note finally that, if $\mathscr{L}$ is any line bundle on X and $D = \Sigma\, n_P P \in \operatorname{Div} X$, then $\mathscr{L} \otimes \mathcal{O}(D)$ can be identified with the sheaf of germs of meromorphic sections σ of $\mathscr{L}$ such that $\operatorname{ord}_P \sigma \geqslant -n_P$.

We conclude this section with the following consequence of the Leray covering theorem ([3], p. 189 or [2], p. 44).

(2.16) PROPOSITION. *Let $f: X \to Y$ be a nonconstant holomorphic map of compact Riemann surfaces, and $\mathscr{V}$ a vector bundle on X. Then the natural maps $H^i(Y, f_0(\mathscr{V})) \to H^i(X, \mathscr{V})$ are isomorphisms for all $i \geqslant 0$.*

Proof: If $\mathfrak{U}$ is a sufficiently fine open covering of Y, then it is clear that, for each $U \in \mathfrak{U}$, $f_0(\mathscr{V}) \mid U$ is $\mathcal{O}_Y$-free, and that $f^{-1}(U)$ is a finite disjoint union of coordinate open sets in X, restricted to each of which $\mathscr{V}$ is free. Since, for $i > 0$, $H^i(W, \mathcal{O}_W) = 0$ for any open $W \subset \mathbf{C}$, it follows that $\mathfrak{U}$ and $\mathfrak{U}' = \{f^{-1}(U) : U \in \mathfrak{U}\}$ are Leray coverings for $f_0(\mathscr{V})$ and $\mathscr{V}$ respectively. Now the natural maps $H^i(\mathfrak{U}, f_0(\mathscr{V})) \to H^i(\mathfrak{U}', \mathscr{V})$ are obviously bijective, q.e.d.

(2.17) *Remark.* Propositions (2.4) and (2.16) are valid (with the same proofs) even if X is not compact, provided we assume that f is *proper*.

(2.18) *Remark.* We know by (2.10) that any (compact) X admits a nonconstant meromorphic function, i.e. a nonconstant holomorphic map $f: X \to \mathbf{P}^1$. Since $\mathbf{P}^1$ is covered by *two* coordinate neighbourhoods which (by (2.11) and (2.12)) constitute a Leray covering for any vector bundle on $\mathbf{P}^1$, it follows by (2.16) that $H^i(X, \mathscr{V}) = 0$ for $i \geqslant 2$ for any compact Riemann surface X and any vector bundle $\mathscr{V}$ on it. This proof is valid in the algebraic situation also. This is the reason for including the case $i \geqslant 2$ in (2.16) rather than appealing to (2.8). We also remark that the Leray theorem is almost trivial for H^1; the fact that for a Leray covering $\mathfrak{U}$, $H^2(\mathfrak{U}, \mathscr{F}) \to H^2(X, \mathscr{F})$ is surjective (which is what was needed above) is also trivial if we use resolutions.

§ 3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

We fix a compact Riemann surface X.

(3.1) *Notation—Definition.* For any vector bundle $\mathscr{V}$ on X, we set

$$h^i(\mathscr{V}) = \dim_{\mathbf{C}} H^i(X, \mathscr{V}),\ i = 0, 1 \text{ and } \chi(\mathscr{V}) = h^0(\mathscr{V}) - h^1(\mathscr{V}).$$

The *genus* g of X is $h^1(\mathcal{O}_X)$.

(3.2) *Remark.* If $0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0$ is an exact sequence of vector bundles, then $\chi(\mathscr{V}) = \chi(\mathscr{V}') + \chi(\mathscr{V}'')$, as follows from the cohomology exact sequence (since $H^2 = 0$).

(3.3) *Definition.* The *degree* deg D of $D = \Sigma\, n(P)\, P \in \operatorname{Div} X$ is $\Sigma\, n(P)$.

(3.4) PROPOSITION. *For any* $D \in \operatorname{Div}(X)$,

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D = \deg D - g + 1 .$$

Proof: (Serre [5], pp. 20-21). The assertion is a tautology for $D = 0$; hence we need only prove that it holds for $D \in \operatorname{Div}(X)$ iff it holds for a divisor of the form $D' = D + P, P \in X$. Now $\mathcal{O}(D)$ is a subsheaf of $\mathcal{O}(D')$, and the quotient sheaf $\mathcal{Q} = \mathcal{O}(D')/\mathcal{O}(D)$ is concentrated at P with stalk isomorphic to $\mathcal{O}_P/\mathfrak{m}_P$. Hence $h^0(\mathcal{Q}) = 1$, and $h^1(\mathcal{Q}) = 0$. Now the exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D') \to \mathcal{Q} \to 0$$

yields the exact sequence

$$0 \to H^0(X, \mathcal{O}(D)) \to \dots \to H^0(X, \mathcal{Q}) \to H^1(X, \mathcal{O}(D)) \\ \to H^1(X, \mathcal{O}(D')) \to 0 ,$$

so that $\chi(\mathcal{O}(D')) - \chi(\mathcal{O}(D)) = 1$. Since $\deg D' - \deg D = 1$, the desired assertion follows, q.e.d.

(3.5) COROLLARY. $h^0(D) \geqslant \deg D - g + 1$.

(3.6) COROLLARY. *For any* $P \in X$, *there exists a nonconstant meromorphic function on* X, *holomorphic in* $X - P$, *with a pole of order* $\leqslant g + 1$ *at* P.

Proof: For $D = (g+1)P, h^0(D) \geqslant 2$ by (3.4), i.e. $H^0(X, \mathcal{O}(D))$ contains a nonconstant element.

(3.7) COROLLARY. *For any vector bundle* $\mathscr{V}$ *on* X, *and any* $P \in X$, $H^1(X - \{P\}, \mathscr{V}) = 0$.

Proof: By (3.6), there exists a holomorphic map $f : X \to \mathbf{P}^1$ with $P = f^{-1}(\infty)$. Now use (2.11), (2.12), (2.16) and (2.17).

(3.8) COROLLARY. $g = 0$ *iff* $X \approx \mathbf{P}^1$.

Proof: $g = 0$ for $X = \mathbf{P}^1$ by Laurent's theorem. Conversely, if $g = 0$, then there exists by (3.6) a meromorphic function f on X with just one *simple* pole and no other singularities. It is easy to see that $f : X \to \mathbf{P}^1$ is then an isomorphism.

(3.9) Corollary. *If* $D \sim D'$, *then* $\deg D = \deg D'$.

Proof: $D \sim D'$ implies $\mathcal{O}(D) \approx \mathcal{O}(D')$, hence $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$. Hence $\deg D = \deg D'$ by (3.4).

(3.10) *Definition.* The *degree* of a line bundle $\mathcal{L}$ is the degree of any $D \in \operatorname{Div} X$ such that $\mathcal{L} \approx \mathcal{O}(D)$, i.e. the degree of the divisor of any meromorphic section of $\mathcal{L}$.

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map $\deg : \operatorname{Pic} X \to \mathbf{Z}$ is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle $\mathcal{V}$ is that of the line bundle $\det \mathcal{V} = \overset{r}{\wedge}_{\mathcal{O}_X} \mathcal{V}$, $r = \operatorname{rank} \mathcal{V}$.

(3.14) *Remark.* The stalk of $(\det \mathcal{V})^{-1} = \operatorname{Hom}(\det \mathcal{V}, \mathcal{O}_X)$ at any $P \in X$ consists $\mathcal{O}_P$-multilinear alternate maps $\mathcal{V}_P \times \ldots \times \mathcal{V}_P$ (r times) $\to \mathcal{O}_P$.

(3.15) Proposition. *If* $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$ *is an exact sequence of vector bundles, then* $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$.

Proof: $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$.

(3.16) Proposition. (Riemann-Roch theorem, preliminary form). *For any vector bundle* $\mathcal{V}$ *on* X,

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \operatorname{rank} \mathcal{V} \, . \, \chi(\mathcal{O})$$

Proof: In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on rank $\mathcal{V}$.

§ 4. The degree of the canonical line bundle

Recall that the canonical line bundle K on X is the sheaf of holomorphic 1-forms.

(4.1) Theorem. $\deg K = 2g - 2 = -2\chi(\mathcal{O})$.

Proof: Choose any nonconstant meromorphic function for X, and consider the holomorphic map $f: X \to \mathbf{P}^1 = Y$. Then the *ramification divisor* $R = \Sigma\, e(P)\, P$ of f is defined as follows: for suitable uniformising parameters z and w at P and $f(P)$ respectively, $w(f(z)) = z^{e(P)+1}$. After composing f with a fractional linear transformation if necessary, we may assume that f is unramified over ∞, i.e. $e(P) = 0$ if $f(P) = \infty$. Note that $r = \sum_{P \in f^{-1}(Q)} (e(P)+1)$ is independent of $Q \in Y$, being clearly the rank of the vector bundle $f_0(\mathcal{O}_X)$ on Y (cf. (2.4)). Now df is a meromorphic 1-form on X (i.e. a meromorphic section of K_X), with zeros of orders $e(P)$ at the P with $f(P) \neq \infty$, and poles of order two at each of the r poles of f. Thus we have:

(4.2) *(Riemann-Hurwitz formula).* $\deg K = \deg R - 2r$.

On the other hand, by (2.16) and (3.16), we have

$$(4.3) \qquad \chi(\mathcal{O}_X) = \chi(f_0(\mathcal{O}_X)) = \deg f_0(\mathcal{O}_X) + r\,\chi(\mathcal{O}_Y)$$
$$= \deg f_0(\mathcal{O}_X) + r\,.$$

Thus, to finish the proof of (4.1), we must prove:

$$(4.4) \qquad \deg f_0(\mathcal{O}_X) = -\frac{1}{2}\deg R\,.$$

To prove (4.4), let $\mathcal{L} = \det f_0(\mathcal{O}_X)$. Then we shall show that there is a canonical $\mathcal{O}_Y$-linear map $\delta : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_Y$ which, at any $Q \in Y$, looks like multiplication by $t_Q^{\delta(Q)}$, where $\delta(Q) = \sum_{P \in f^{-1}(Q)} e(P)$ (t_Q a uniformising parameter at Q). Since $\sum_Q \delta(Q) = \deg R$, this will prove (4.4).

The map δ is the classical discriminant map. To define it, we first define the "trace" map $\tau : f_0(\mathcal{O}_X) \to \mathcal{O}_Y$: for $U \subset Y$ open and $h \in \mathcal{O}_X(f^{-1}(U))$, $\tau(h)(Q) = \sum_{P \in f^{-1}(Q)} (e(P)+1)\, h(P)$ for all $Q \in U$. Then clearly $\tau(h) \in \mathcal{O}_Y(U)$, and τ is $\mathcal{O}_Y$-linear. Now for any $U \subset Y$ open and any two r-tuples $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_r)$ of elements of $\mathcal{O}_X(f^{-1}(U))$ (recall that $r = \operatorname{rank} f_0(\mathcal{O}_X)$), we set $\delta(\lambda, \mu) = \det(\tau(\lambda_i \mu_j))$. Clearly δ is $\mathcal{O}_Y$-multilinear and alternating in each of λ and μ, hence defines an $\mathcal{O}_X$-linear map

$$\delta : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_Y$$

This is the desired map. To compute the effect of δ at any $Q \in Y$, let us assume first that $f^{-1}(Q)$ is a single point P. In suitable coordinate systems at P and Q, f is the map $Z \to Z^{e_p+1} = w$ of the unit disc $U \subset \mathbf{C}$ onto another copy W of it. Since $1, Z, \ldots, Z^{e_p}$ provide an $\mathcal{O}_W$-basis for $f_0(Q_U)$, the value of δ on a local generator of $\mathcal{L} \otimes \mathcal{L}$ is given by

$$\det\left(\tau\left(Z^{i+j}\right)\right), \; 0 \leqslant i, j \leqslant e = e_p .$$

But

$$\tau\left(Z^{i+j}\right) = Z^{i+j}\left(1 + \zeta^{i+j} + (\zeta^{i+j})^2 + \ldots + (\zeta^{i+j})^e\right),$$

(ζ denoting a primitive $(e+1) - st$ root of unity), hence

$$\begin{aligned}\tau\left(Z^{i+j}\right) &= (e+1)\,Z^{i+j} \quad \text{if} \quad i+j = 0 \quad \text{or} \quad e+1, \\ &= 0 \quad \text{otherwise}.\end{aligned}$$

Hence $\det\left(\tau\left(Z^{i+j}\right)\right)$ is a (nonzero) constant multiple of $Z^{(e+1)e} = w^e$ as asserted.

If $f^{-1}(Q)$ consists of several points, the situation is a direct sum of those considered above, and δ is indeed as asserted. This proves Theorem (4.1).

(4.5) *Remark.* Let the notation be as above, and let $E(X)$ denote the topological Euler-Poincaré characteristic of X. Then, using the formula $E(X) =$ number of vertices $-$ number of edges $+$ number of faces in any triangulation of X, it is easy to see that $E(X) = r\,E(Y) - \deg R\,(Y{=}\mathbf{P}^1)$. Indeed, choose any triangulation of Y which contains all the images of the ramification points of f as vertices, and lift it to a triangulation of X. Then, while r edges or faces lie over each edge or face of Y, the ramification points reduce the number of vertices over certain vertices of Y, and one gets the formula asserted. Since $E(Y) = 2$, (4.2) yields:

(4.6) COROLLARY. $\deg K_X = -E(X) = 2g - 2$, *i.e.* g *is also the topological genus* $(1/2)\,b_1(X)$ *of the compact oriented surface* X.

§ 5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

(5.1) (RIEMANN-ROCH THEOREM). For any line bundle $\mathcal{L}$ on X,

$$h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1 .$$

Proof: It is enough to prove

(5.2) for all $\mathscr{L}$, $h^0(\mathscr{L}) - h^0(K \otimes \mathscr{L}^{-1}) \geqslant \deg \mathscr{L} - g + 1$. For then, replacing $\mathscr{L}$ by $K \otimes \mathscr{L}^{-1}$ changes only the sign of the left side, and the *same is true of the right side by* (4.1) (cf. [4], p. 147).

Now (5.2) is true if $\deg \mathscr{L} > \deg K$, for then $h^0(K \otimes \mathscr{L}^{-1}) = 0$, and we can use (3.5). Thus, to prove (5.2), we may assume that $\mathscr{L} = \mathcal{O}(D)$ for some $D \in \operatorname{Div} X$, and that (5.2) holds for $\mathscr{L}' = \mathcal{O}(D+P_0)$, $P_0 \in X$. Now it is clear that $h^0(\mathscr{L}') \leqslant h^0(\mathscr{L}) + 1$, and similarly $h^0(K \otimes \mathscr{L}'^{-1}) \leqslant h^0(K \otimes \mathscr{L}') + 1$ (cf. the proof of (3.4)). So (5.2) fails for $\mathscr{L}$ if and only if (*) $h^0(\mathscr{L}') = h^0(\mathscr{L}) + 1$, *and* $h^0(K \otimes \mathscr{L}^{-1}) = h^0(K \otimes \mathscr{L}'^{-1}) + 1$. But if (*) holds, there exist

$$\sigma \in H^0(X, \mathscr{L}') - H^0(X, \mathscr{L})$$

and

$$\omega \in H^0(X, K \otimes \mathscr{L}^{-1}) - H^0(X, K \otimes \mathscr{L}'^{-1}),$$

and then

$$\sigma\omega = \sigma \otimes \omega \in H^0(X, K \otimes \mathcal{O}(P_0)) - H^0(X, K),$$

i.e. $\sigma\omega$ is a meromorphic form with precisely one simple pole at P_0. But this is impossible: if D is a disc around P_0 in some coordinate system centred at P_0, then $\int_{\partial D} \sigma\omega = - \int_{\partial(X-D)} \sigma\omega = 0$ by Stokes' theorem, while $\int_{\partial D} \sigma\omega \neq 0$ by the Residue theorem. Thus (*) cannot hold, and (5.2) is proved, q.e.d.

(5.3) Corollary. *For any line bundle* $\mathscr{L}$ *on* X, $h^1(\mathscr{L}) = h^0(K \otimes \mathscr{L}^{-1})$.

Proof: Compare (5.1) and (3.4).

(5.4) Corollary. $h^0(K) = g$ *and* $h^1(K) = 1$.

Before proceeding to Serre duality, we examine the notion of residue in greater detail. Thus let $U \subset X$ be open, and ω a meromorphic 1-form on U with a pole at $P \in U$. Then, in terms of a uniformising parameter t at P, $\omega = f\,dt$ near P, with f a meromorphic function at P. The *residue* of ω at P is $\frac{1}{2\pi i}$ times the coefficient of $1/t$ in the Laurent expansion of f in powers of t. The independence of $\operatorname{Res}_P(\ )$ on the choice of t can be proved either by direct computation or by identifying it with $1/2\pi i \int_\gamma \omega$, where γ is a suitable curve around P. By the argument already used above (Stokes' theorem), one gets

(5.5) (Residue theorem). *The sum of the residues of any meromorphic 1-form on* X *is zero.*

(5.6) Corollary. *Given distinct* $P, Q \in X$, *there exists a meromorphic 1-form on* X, *holomorphic outside* P *and* Q, *and with simple poles at* P, Q *of residue* 1 *and* -1 *respectively.*

Proof: Let $\mathscr{L} = K \otimes \mathcal{O}(P+Q)$. Then $\deg K \otimes \mathscr{L}^{-1} < 0$, hence $h^0(\mathscr{L}) = g+1$ by (5.1), i.e. there exists $\omega \in H^0(X, \mathscr{L}) - H^0(X, K)$. Then it is clear that the residues of ω at P and Q must be non-zero, while their sum is zero (by (5.5)), hence a suitable constant multiple of ω will have the desired properties.

(5.7) Proposition. *There is a* canonical *isomorphism* res : $H^1(X, K) \to \mathbf{C}$.

Proof: Pick any $P \in X$, and a coordinate neighbourhood U of P. Let $\mathfrak{U}$ be the covering $\{U, X-P\}$ of X. Then, by taking residues at P, we get a map $\operatorname{res}_P : Z^1(\mathfrak{U}, K) \to \mathbf{C}$. This map is not zero, and induces a map $H^1(\mathfrak{U}, K) \to C$ (by the residue theorem). Since $h^1(K) = 1$, $\operatorname{res}_P : H^1(\mathfrak{U}, K) \to H^1(X, K) \to \mathbf{C}$ is in fact an isomorphism. That the map $\operatorname{res}_P : H^1(X, K) \to \mathbf{C}$ is independent of the choice of $P \in X$ is precisely the meaning of (5.6), and we get the asserted canonical isomorphism res.

(5.8) Serre Duality. *For any line bundle* $\mathscr{L}$ *on* X, *the natural bilinear form*

$$\zeta : H^0(X, \mathscr{L}) \times H^1(X, K \otimes \mathscr{L}^{-1}) \to H^1(X, K) \xrightarrow{res} \mathbf{C}$$

is nondegenerate.

(5.9) *Remark.* For any covering $\mathfrak{U}$ of X, the natural map $\mathscr{L} \times (K \otimes \mathscr{L}^{-1}) \to K$ defines an obvious pairing

$$H^0(X, \mathscr{L}) \times Z^1(\mathfrak{U}, K \otimes \mathscr{L}^{-1}) \to Z^1(\mathfrak{U}, K)$$

which is easily seen to induce the pairing

$$H^0(X, \mathscr{L}) \times H^1(X, K \otimes \mathscr{L}^{-1}) \to H^1(X, K)$$

figuring in (5.8).

Proof of (5.8). Since we already know that

$$h^0(X, \mathscr{L}) = h^1(X, K \otimes \mathscr{L}^{-1}),$$

we need only show that, if $\sigma \in H^0(X, \mathscr{L})$ is such that $\zeta(\sigma \otimes \gamma) = 0$ for all $\gamma \in H^1(X, K \otimes \mathscr{L}^{-1})$, then $\sigma \equiv 0$. Now choose any $P \in X$, and a co-ordinate neighbourhood (U, z) of P centred at P such that $\mathscr{L} \mid U \approx \mathcal{O}_U$. Then the covering $\mathfrak{U} = \{U, X - P\}$ is a Leray covering for $\mathscr{L}, K$ and $K \otimes \mathscr{L}^{-1}$ ((3.7)). The $z^n\, dz, n \in \mathbf{Z}$, can all be regarded as elements of $Z^1(\mathfrak{U}, K \otimes \mathscr{L}^{-1})$; let γ_n denote their images in $H^1(X, K \otimes \mathscr{L}^{-1})$. Then clearly $\rho(\sigma \otimes \gamma_n) = 0$ for all n implies that all the coefficients of the Taylor expansion of σ at P with respect to vanish, hence $\sigma \equiv 0$, q.e.d.

(5.9) SERRE DUALITY FOR VECTOR BUNDLES. *For any vector bundle $\mathscr{V}$ on X, let $\mathscr{V}^* = \operatorname{Hom} \mathcal{O}_X(\mathscr{V}, \mathcal{O}_X)$. Then the natural pairing*

$$\zeta : H^0(X, \mathscr{V}) \times H^1(X, K \otimes \mathscr{V}^*) \to H^1(X, K) \overset{res}{\tilde{\to}} \mathbf{C}$$

is non-degenerate.

Proof: Arguing as in the proof of (5.8) we see that the map $H^0(X, \mathscr{V}) \to (H^1(X, K \otimes \mathscr{V}^*))^*$ induced by ζ is injective, hence $h^0(X, \mathscr{V}) \leqslant h^1(X, K \otimes \mathscr{V}^*)$. Replacing $\mathscr{V}$ by $K \otimes \mathscr{V}^*$, we also get $h^0(K \otimes \mathscr{V}^*) \leqslant h^1(\mathscr{V})$. But, by induction on rank $\mathscr{V}$, we easily deduce from (5.3) that $\chi(K \otimes \mathscr{V}^*) = -\chi(\mathscr{V})$, hence $h^0(X, \mathscr{V}) = h^1(X, K \otimes \mathscr{V}^*)$. Thus ζ is non-degenerate as before.

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