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THE RIEMANN-ROCH THEOREM FOR COMPACT RIEMANN SURFACES

by R. R. Simha

§ 1. Introduction

The aim of this article is to present a sheaf-theoretic proof of the Riemann-Roch theorem (including Serre duality) for vector bundles on compact Riemann surfaces. The basic assumption will be the finite dimensionality of cohomology vector spaces; no further potential theory will be used. Thus the proof will work (with trivial modifications) in the algebraic case also (over an algebraically closed field of any characteristic). The possibly new contribution of the article is a simple direct proof of the fact that the degree of the canonical divisor is $2g - 2$, where $g = \dim H^1(X, \mathcal{O})$.

We now give an outline of the contents. The rather long Section 2 gives the necessary definitions and sheaf-theoretic results, and the consequences of the finite dimensionality theorem which are needed later. Section 3 gives the preliminary form of the Riemann-Roch theorem. The identity $\deg K = 2g - 2$ is proved in Section 4. Serre duality and the final form of the Riemann-Roch theorem are proved in Section 5.

Our exposition borrows freely from those of Serre [5] and Mumford [4]. We should also mention the proof of the Riemann-Roch theorem given in Grauert-Remmert [1] (Ch. VII).

I thank the referee for his careful reading of the manuscript, which has eliminated many errors.

§ 2. Line bundles and vector bundles. Sheaf theoretic preliminaries

In all that follows, X will denote a compact Riemann surface, i.e. a connected compact complex manifold of complex dimension 1; $\mathcal{O} = \mathcal{O}_X$ will denote its structure sheaf, i.e. the sheaf of germs of holomorphic functions on X.

For any sheaf $\mathscr{F}$ on X, and any $P \in X$, $\mathscr{F}_P$ denotes the stalk of $\mathscr{F}$ at P; for $U \subset X$ open, $\mathscr{F}(U)$ denotes the set of sections of $\mathscr{F}$ over U.

(2.1) *Definition.* A *vector bundle of rank* r on X is an $\mathcal{O}_X$-Module (i.e. a sheaf of $\mathcal{O}_X$-modules) which is locally $\mathcal{O}_X$-isomorphic to $\mathcal{O}_X + \ldots + \mathcal{O}_X$ (r times). A *line bundle* is a vector bundle of rank one.

(2.2) *Example—Definition.* A divisor $D = \sum_{P \in X} n(P)P$ on X is just an element of the free abelian group Div (X) on the set X. We write $D \geqslant 0$ if $n(P) \geqslant 0$ for all $P \in X$, and $D \geqslant D'$ for another $D' \in \text{Div}(X)$ if $D - D' \geqslant 0$. For any such $D \in \text{Div}(X)$, we define a line bundle $\mathcal{O}(D)$ as follows. Let $\mathscr{M} = \mathscr{M}_X$ be the sheaf of germs of meromorphic functions on X. Then, for any $U \subset X$ open, $\mathcal{O}(D)(U) = \{f \in \mathscr{M}(U) : \text{ord}_P f \geqslant -n(P) \text{ for all } P \in U\}$. Then, for any $P \in X$, it is clear that $t_P^{-n(P)}$ is a local generator for $\mathcal{O}(D)$ near P, where t_P is a uniformising parameter at P; thus $\mathcal{O}(D)$ is indeed a line bundle, which is an $\mathcal{O}$-submodule of $\mathscr{M}$. It is clear that $D \geqslant 0$ iff $\mathcal{O}(D) \supset \mathcal{O}$.

(2.3) *Example.* The *canonical line bundle* $K_X = K$ on X is just the sheaf of holomorphic 1-forms on X. Thus, if (U, z) is a coordinate chart on X, then $K(U)$ is the set of differential 1-forms $f\,dz$ on U with $f \in \mathcal{O}(U)$, so that K is clearly a line bundle.

(2.4) *Example—Proposition.* Let $f : X \to Y$ be a nonconstant holomorphic map of compact Riemann surfaces, and $\mathscr{V}$ a vector bundle on X. Then the direct image sheaf $f_0(\mathscr{V})$ of $\mathscr{V}$ by f is a vector bundle on Y.

Proof: Recall that, for any $U \subset Y$ open, $f_0(\mathscr{V})(U) = \mathscr{V}(f^{-1}(U))$ and that $g \in \mathcal{O}_Y(U)$ acts as multiplication by $g \circ f$. Now note that f is a *proper* map, and that $f^{-1}(Q)$ is a finite set for each $Q \in Y$. Also, for any $P \in X$, there exist uniformising parameters z and w at P and $f(P)$ respectively such that $w \circ f = z^n$ for some integer $n \geqslant 1$. Thus it is easily seen that it suffices to prove the following: for the map $f : z \to z^n$ of the unit disc U in $\mathbf{C}$ onto another copy W of it, $f_0(\mathcal{O}_U)$ is a free $\mathcal{O}_W$-Module (or rank n). But this is clear; in fact the functions $1, z, \ldots, z^{n-1}$, regarded as sections of $f_0(\mathcal{O}_V)$ over W, generate it over $\mathcal{O}_W$ and are independent everywhere on W.

(2.5) *Definition.* A *meromorphic section* σ of a vector bundle $\mathscr{V}$ on X is a holomorphic section σ of $\mathscr{V}$ over the complement of some finite set $S \subset X$ such that, for each $P \in S$, there exists a connected neighbourhood

U of P and an $f \not\equiv 0$ in $\mathcal{O}(U)$ so that $f\sigma$ extends to a holomorphic section of V over U.

(2.6) *Remark.* A meromorphic section of $\mathcal{O}_X$ is just a meromorphic function on X. It is clear that the set of meromorphic sections of a vector bundle $\mathscr{V}$ is a vector space over the field of meromorphic functions on X, of dimension $\leqslant 1$ if rank $\mathscr{V} = 1$.

(2.7) *Definition.* The *divisor* $\operatorname{div} \sigma$ *of a meromorphic section* $\sigma \not\equiv 0$ of a vector bundle $\mathscr{V}$ on X is $\Sigma\, n(P)\, P$, where, for each $P \in X$, $n(P)$ is the integer characterised by $t_P^{-n(P)} \sigma \in \mathscr{V}_P - \mathfrak{m}_P \mathscr{V}_P$; here t_P is a uniformising parameter at P, and $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_P$; $n(P)$ is the *order* of σ at P.

We shall now deduce from the finiteness theorem that every vector bundle has plenty of meromorphic sections. We first state the finiteness theorem explicitly:

(2.8) *Finiteness Theorem. For every vector bundle* $\mathscr{V}$ *on* X, $H^0(X, \mathscr{V})$ *and* $H^1(X, \mathscr{V})$ *are finite-dimensional vector spaces over* $\mathbf{C}$;

$$H^2(X, \mathscr{V}) = 0\,.$$

(2.9) *Remark.* The finite dimensionality of H^0 and H^1 can be deduced from Montel's theorem and the fact that a locally compact Hilbert space is finite dimensional; see e.g. Gunning [2], p. 59 or [1], Ch. VI. The vanishing of H^i for $i \geqslant 2$ follows from the Dolbeault resolution for $\mathscr{V}$, see e.g. Gunning-Rossi [3], pp. 184; another proof will be indicated in (2.17).

(2.10) PROPOSITION. *Every vector bundle* $\mathscr{V}$ *on* X *admits (infinitely many) meromorphic sections.*

Proof: Pick any $P \in X$, and let (U, z) be a coordinate system centred at P (i.e. $z(P) = 0$). Let $\mathfrak{U}$ be the covering $\{U, X - P\}$ of X. We may assume that there is an $\mathcal{O}_U$-isomorphism $\varphi : \mathcal{O}_U^r \to \mathscr{V} \mid U$ ($r = \operatorname{rank} \mathscr{V}$). Then the set of r-tuples of polynomials in $1/z$ can be regarded (via φ) as an infinite dimensional subspace W of $Z^1(\mathfrak{U}, \mathscr{V})$. Now $H^1(\mathfrak{U}, \mathscr{V}) \hookrightarrow H^1(X, \mathscr{V})$ is finite dimensional, hence the kernel W' of the natural map $W \hookrightarrow Z^1(\mathfrak{U}, \mathscr{V}) \to H^1(\mathfrak{U}, \mathscr{V})$ is infinite dimensional. It is clear that different elements of W' lead to different meromorphic sections of $\mathscr{V}$,

q.e.d.

(2.11) COROLLARY. *Every vector bundle* $\mathscr{V}$ *on* X *is an extension of line bundles of the form* $\mathcal{O}(D)$, $D \in \operatorname{Div} X$ *(i.e. there exists an exact sequence*

$0 \to \mathcal{O}(D) \to \mathcal{V} \to \mathcal{V}' \to 0$ *with* $\mathcal{V}'$ *a vector bundle); every line bundle* $\mathcal{L}$ *is isomorphic to* $\mathcal{O}$ (div σ) *for any meromorphic section* σ *of* $\mathcal{L}$.

Proof: Choose any meromorphic section $\sigma\,(\not\equiv 0)$ of $\mathcal{V}$, and let $D = \operatorname{div}\sigma$. Then multiplication by σ makes $\mathcal{O}(D)$ an $\mathcal{O}_X$-submodule of $\mathcal{V}$, and it is clear that $\mathcal{V} \,|\, \mathcal{O}(D)$ is again a vector bundle, q.e.d.

(2.12) Corollary. *Let* $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ *be the Riemann sphere, and* $\mathcal{L}$ *a line bundle on* $\mathbf{P}^1$. *Then* $\mathcal{L} \,|\, \mathbf{C}$ *is trivial, i.e.* $\mathcal{L} \,|\, \mathbf{C} \approx \mathcal{O}_{\mathbf{C}}$.

Proof: Let σ be a meromorphic section of $\mathcal{L}$ over $\mathbf{P}^1$, and $\operatorname{div}\sigma = \Sigma\, n(P)\, P$. If z denotes the coordinate function on $\mathbf{C}$, then clearly $\sigma' = \prod_{P \in \mathbf{C}} (z - z(P))^{-n(P)}$. σ is a nowhere-vanishing section of $\mathcal{L}$ over $\mathbf{C}$, q.e.d.

In order to see when, for $D, D' \in \operatorname{Div}(X)$, $\mathcal{O}(D)$ and $\mathcal{O}(D')$ are isomorphic as line bundles, we begin with a definition:

(2.13) *Definition.* Let $D, D' \in \operatorname{Div} X$. Then D is *equivalent* to D' (notation: $D \sim D'$) if there exists a meromorphic function $f \not\equiv 0$ such that $D' = D + \operatorname{div} f$.

Remark. An f as in (2.13), if it exists, is clearly unique upto a nonzero constant factor (X compact!)

(2.14) Proposition. *Let* $D, D' \in \operatorname{Div}(X)$. *Then* $D \sim D'$ *iff* $\mathcal{O}(D)$ *and* $\mathcal{O}(D')$ *are isomorphic.*

Proof: Note that any $\mathcal{O}_X$-linear map $\mathcal{M}_X \to \mathcal{M}_X$ is defined by multiplication by a unique meromorphic function, and that any $\mathcal{O}_X$-linear map $\mathcal{O}(D) \to \mathcal{O}(D')$ extends naturally to one of $\mathcal{M}_X$ into itself. Now multiplication by the meromorphic function f maps $\mathcal{O}(D)$ into $\mathcal{O}(D')$ iff $-D + \operatorname{div} f \geqslant -D'$, i.e. $D' + \operatorname{div} f \geqslant D$, so the proposition follows.

(2.15) *Remark.* The map $D \to \mathcal{O}(D)$ thus sets up a bijection between the set of equivalence classes of divisors and the set Pic X of isomorphism classes of line bundles on X. Now the group structure on Div X clearly induces one on the set of divisor classes, hence by the above map on Pic X. It is easy to see that the induced group operation on Pic X corresponds to the tensor product (over $\mathcal{O}_X$) of line bundles: multiplication of functions induces a canonical map $\mathcal{O}(D) \otimes \mathcal{O}(D') \to \mathcal{O}(D + D')$ which is clearly an isomorphism. It is also easy to verify that the inverse of the line bundle $\mathcal{L}$ is represented by $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ (cf. the proof of (2.14)).

Note finally that, if $\mathscr{L}$ is any line bundle on X and $D = \Sigma n_P P \in \operatorname{Div} X$, then $\mathscr{L} \otimes \mathcal{O}(D)$ can be identified with the sheaf of germs of meromorphic sections σ of $\mathscr{L}$ such that $\operatorname{ord}_P \sigma \geqslant -n_P$.

We conclude this section with the following consequence of the Leray covering theorem ([3], p. 189 or [2], p. 44).

(2.16) PROPOSITION. *Let* $f : X \to Y$ *be a nonconstant holomorphic map of compact Riemann surfaces, and* $\mathscr{V}$ *a vector bundle on* X. *Then the natural maps* $H^i(Y, f_0(\mathscr{V})) \to H^i(X, \mathscr{V})$ *are isomorphisms for all* $i \geqslant 0$.

Proof: If $\mathfrak{U}$ is a sufficiently fine open covering of Y, then it is clear that, for each $U \in \mathfrak{U}$, $f_0(\mathscr{V}) \mid U$ is $\mathcal{O}_Y$-free, and that $f^{-1}(U)$ is a finite disjoint union of coordinate open sets in X, restricted to each of which $\mathscr{V}$ is free. Since, for $i > 0$, $H^i(W, \mathcal{O}_W) = 0$ for any open $W \subset \mathbf{C}$, it follows that $\mathfrak{U}$ and $\mathfrak{U}' = \{f^{-1}(U) : U \in \mathfrak{U}\}$ are Leray coverings for $f_0(\mathscr{V})$ and $\mathscr{V}$ respectively. Now the natural maps $H^i(\mathfrak{U}, f_0(\mathscr{V})) \to H^i(\mathfrak{U}', \mathscr{V})$ are obviously bijective, q.e.d.

(2.17) *Remark.* Propositions (2.4) and (2.16) are valid (with the same proofs) even if X is not compact, provided we assume that f is *proper*.

(2.18) *Remark.* We know by (2.10) that any (compact) X admits a non-constant meromorphic function, i.e. a nonconstant holomorphic map $f : X \to \mathbf{P}^1$. Since $\mathbf{P}^1$ is covered by *two* coordinate neighbourhoods which (by (2.11) and (2.12)) constitute a Leray covering for any vector bundle on $\mathbf{P}^1$, it follows by (2.16) that $H^i(X, \mathscr{V}) = 0$ for $i \geqslant 2$ for any compact Riemann surface X and any vector bundle $\mathscr{V}$ on it. This proof is valid in the algebraic situation also. This is the reason for including the case $i \geqslant 2$ in (2.16) rather than appealing to (2.8). We also remark that the Leray theorem is almost trivial for H^1; the fact that for a Leray covering $\mathfrak{U}$, $H^2(\mathfrak{U}, \mathscr{F}) \to H^2(X, \mathscr{F})$ is surjective (which is what was needed above) is also trivial if we use resolutions.

§ 3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

We fix a compact Riemann surface X.

(3.1) *Notation—Definition.* For any vector bundle $\mathscr{V}$ on X, we set

$$h^i(\mathscr{V}) = \dim_{\mathbf{C}} H^i(X, \mathscr{V}),\ i = 0, 1 \text{ and } \chi(\mathscr{V}) = h^0(\mathscr{V}) - h^1(\mathscr{V}).$$

The *genus* g of X is $h^1(\mathcal{O}_X)$.