

# §3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **27 (1981)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.08.2024**

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Note finally that, if  $\mathcal{L}$  is any line bundle on  $X$  and  $D = \sum n_p P \in \text{Div } X$ , then  $\mathcal{L} \otimes \mathcal{O}(D)$  can be identified with the sheaf of germs of meromorphic sections  $\sigma$  of  $\mathcal{L}$  such that  $\text{ord}_p \sigma \geq -n_p$ .

We conclude this section with the following consequence of the Leray covering theorem ([3], p. 189 or [2], p. 44).

(2.16) PROPOSITION. *Let  $f: X \rightarrow Y$  be a nonconstant holomorphic map of compact Riemann surfaces, and  $\mathcal{V}$  a vector bundle on  $X$ . Then the natural maps  $H^i(Y, f_0(\mathcal{V})) \rightarrow H^i(X, \mathcal{V})$  are isomorphisms for all  $i \geq 0$ .*

*Proof:* If  $\mathcal{U}$  is a sufficiently fine open covering of  $Y$ , then it is clear that, for each  $U \in \mathcal{U}$ ,  $f_0(\mathcal{V})|_U$  is  $\mathcal{O}_Y$ -free, and that  $f^{-1}(U)$  is a finite disjoint union of coordinate open sets in  $X$ , restricted to each of which  $\mathcal{V}$  is free. Since, for  $i > 0$ ,  $H^i(W, \mathcal{O}_W) = 0$  for any open  $W \subset \mathbb{C}$ , it follows that  $\mathcal{U}$  and  $\mathcal{U}' = \{f^{-1}(U) : U \in \mathcal{U}\}$  are Leray coverings for  $f_0(\mathcal{V})$  and  $\mathcal{V}$  respectively. Now the natural maps  $H^i(\mathcal{U}, f_0(\mathcal{V})) \rightarrow H^i(\mathcal{U}', \mathcal{V})$  are obviously bijective, q.e.d.

(2.17) Remark. Propositions (2.4) and (2.16) are valid (with the same proofs) even if  $X$  is not compact, provided we assume that  $f$  is proper.

(2.18) Remark. We know by (2.10) that any (compact)  $X$  admits a nonconstant meromorphic function, i.e. a nonconstant holomorphic map  $f: X \rightarrow \mathbb{P}^1$ . Since  $\mathbb{P}^1$  is covered by two coordinate neighbourhoods which (by (2.11) and (2.12)) constitute a Leray covering for any vector bundle on  $\mathbb{P}^1$ , it follows by (2.16) that  $H^i(X, \mathcal{V}) = 0$  for  $i \geq 2$  for any compact Riemann surface  $X$  and any vector bundle  $\mathcal{V}$  on it. This proof is valid in the algebraic situation also. This is the reason for including the case  $i \geq 2$  in (2.16) rather than appealing to (2.8). We also remark that the Leray theorem is almost trivial for  $H^1$ ; the fact that for a Leray covering  $\mathcal{U}$ ,  $H^2(\mathcal{U}, \mathcal{F}) \rightarrow H^2(X, \mathcal{F})$  is surjective (which is what was needed above) is also trivial if we use resolutions.

### § 3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

We fix a compact Riemann surface  $X$ .

(3.1) Notation—Definition. For any vector bundle  $\mathcal{V}$  on  $X$ , we set

$$h^i(\mathcal{V}) = \dim_{\mathbb{C}} H^i(X, \mathcal{V}), \quad i = 0, 1 \quad \text{and} \quad \chi(\mathcal{V}) = h^0(\mathcal{V}) - h^1(\mathcal{V}).$$

The genus  $g$  of  $X$  is  $h^1(\mathcal{O}_X)$ .

(3.2) *Remark.* If  $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$  is an exact sequence of vector bundles, then  $\chi(\mathcal{V}) = \chi(\mathcal{V}') + \chi(\mathcal{V}'')$ , as follows from the cohomology exact sequence (since  $H^2 = 0$ ).

(3.3) *Definition.* The *degree*  $\deg D$  of  $D = \sum n(P) P \in \text{Div } X$  is  $\sum n(P)$ .

(3.4) **PROPOSITION.** For any  $D \in \text{Div}(X)$ ,

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D = \deg D - g + 1.$$

*Proof:* (Serre [5], pp. 20-21). The assertion is a tautology for  $D = 0$ ; hence we need only prove that it holds for  $D \in \text{Div}(X)$  iff it holds for a divisor of the form  $D' = D + P, P \in X$ . Now  $\mathcal{O}(D)$  is a subsheaf of  $\mathcal{O}(D')$ , and the quotient sheaf  $\mathcal{Q} = \mathcal{O}(D')/\mathcal{O}(D)$  is concentrated at  $P$  with stalk isomorphic to  $\mathcal{O}_P/\mathfrak{m}_P$ . Hence  $h^0(\mathcal{Q}) = 1$ , and  $h^1(\mathcal{Q}) = 0$ . Now the exact sequence

$$0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D') \rightarrow \mathcal{Q} \rightarrow 0$$

yields the exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}(D)) \rightarrow \dots \rightarrow H^0(X, \mathcal{Q}) \rightarrow H^1(X, \mathcal{O}(D)) \\ \rightarrow H^1(X, \mathcal{O}(D')) \rightarrow 0, \end{aligned}$$

so that  $\chi(\mathcal{O}(D')) - \chi(\mathcal{O}(D)) = 1$ . Since  $\deg D' - \deg D = 1$ , the desired assertion follows, q.e.d.

(3.5) **COROLLARY.**  $h^0(D) \geq \deg D - g + 1$ .

(3.6) **COROLLARY.** For any  $P \in X$ , there exists a nonconstant meromorphic function on  $X$ , holomorphic in  $X - P$ , with a pole of order  $\leq g + 1$  at  $P$ .

*Proof:* For  $D = (g+1)P$ ,  $h^0(D) \geq 2$  by (3.4), i.e.  $H^0(X, \mathcal{O}(D))$  contains a nonconstant element.

(3.7) **COROLLARY.** For any vector bundle  $\mathcal{V}$  on  $X$ , and any  $P \in X$ ,  $H^1(X - \{P\}, \mathcal{V}) = 0$ .

*Proof:* By (3.6), there exists a holomorphic map  $f: X \rightarrow \mathbf{P}^1$  with  $P = f^{-1}(\infty)$ . Now use (2.11), (2.12), (2.16) and (2.17).

(3.8) **COROLLARY.**  $g = 0$  iff  $X \approx \mathbf{P}^1$ .

*Proof:*  $g = 0$  for  $X = \mathbf{P}^1$  by Laurent's theorem. Conversely, if  $g = 0$ , then there exists by (3.6) a meromorphic function  $f$  on  $X$  with just one

simple pole and no other singularities. It is easy to see that  $f: X \rightarrow \mathbf{P}^1$  is then an isomorphism.

(3.9) COROLLARY. *If  $D \sim D'$ , then  $\deg D = \deg D'$ .*

*Proof:*  $D \sim D'$  implies  $\mathcal{O}(D) \approx \mathcal{O}(D')$ , hence  $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$ . Hence  $\deg D = \deg D'$  by (3.4).

(3.10) *Definition.* The *degree* of a line bundle  $\mathcal{L}$  is the degree of any  $D \in \text{Div } X$  such that  $\mathcal{L} \approx \mathcal{O}(D)$ , i.e. the degree of the divisor of any meromorphic section of  $\mathcal{L}$ .

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map  $\deg: \text{Pic } X \rightarrow \mathbf{Z}$  is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle  $\mathcal{V}$  is that of the line bundle  $\det \mathcal{V} = \bigwedge_{\mathcal{O}_x}^r \mathcal{V}$ ,  $r = \text{rank } \mathcal{V}$ .

(3.14) *Remark.* The stalk of  $(\det \mathcal{V})^{-1} = \text{Hom}(\det \mathcal{V}, \mathcal{O}_X)$  at any  $P \in X$  consists  $\mathcal{O}_P$ -multilinear alternate maps  $\mathcal{V}_P \times \dots \times \mathcal{V}_P$  ( $r$  times)  $\rightarrow \mathcal{O}_P$ .

(3.15) PROPOSITION. *If  $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$  is an exact sequence of vector bundles, then  $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$ .*

*Proof:*  $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$ .

(3.16) PROPOSITION. (Riemann-Roch theorem, preliminary form). *For any vector bundle  $\mathcal{V}$  on  $X$ ,*

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \text{rank } \mathcal{V} \cdot \chi(\mathcal{O})$$

*Proof:* In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on rank  $\mathcal{V}$ .

#### § 4. THE DEGREE OF THE CANONICAL LINE BUNDLE

Recall that the canonical line bundle  $K$  on  $X$  is the sheaf of holomorphic 1-forms.

(4.1) THEOREM.  $\deg K = 2g - 2 = -2 \chi(\mathcal{O})$ .