

# 5. A NON-EXISTENT HIERARCHY

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complete when so operated on. A most convenient starting point is the following family  $T$  which is of  $p$ -bounded formula size:

$$T_{n^2+n} = \prod_{k=1}^n \sum_{i=1}^n x_{k,i} y_i .$$

Clearly (i) the coefficient of  $y_1 \dots y_n$  in  $T_{n^2+n}$ ,

$$(ii) \quad \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} \dots \frac{\partial}{\partial y_n} T_{n^2+n} , \text{ and}$$

$$(iii) \quad \left(\frac{3}{2}\right)^n \int_{-1}^1 \dots \int_{-1}^1 [y_1 \dots y_n T_{n^2+n}] dy_1 \dots dy_n$$

all equal  $\text{Perm} \{ x_{k,i} \}$ .

In contrast, it is easy to see that all the other operations that we have considered preserve  $p$ -computability. This is immediate in the case of substitution. It can be shown to be true for  $\partial P / \partial x_i$  and  $\int P dx_i$  by considering a program for  $P$ , and decomposing it according to the powers of  $x_i$  at each instruction in the manner of [12].

## 5. A NON-EXISTENT HIERARCHY

By analogies with recursion theory we can attempt to define the following hierarchy:

*Definition.*  $PD^0$  = class of  $p$ -computable polynomial families. For  $i > 0$   $P \in PD^i$  iff  $P$  is defined by some  $Q \in PD^{i-1}$  in the sense of Definition 3.

That this hierarchy collapses in this algebraic case is easy to see:

**THEOREM 5.** For any  $F$  and any  $i > 0$   $PD^i = PD^{i+1}$ .

*Proof.* It is clearly sufficient to prove  $PD^1 = PD^2$ . If  $P \in PD^2$  then for each  $m$

$$P_m(\mathbf{x}) = \sum_{\mathbf{b}} Q_i(\mathbf{x}, \mathbf{b})$$

where for some  $R \in PD^0$  for each  $i$

$$Q_i(\mathbf{x}, \mathbf{b}) = \sum_{\mathbf{c}} R_j(\mathbf{x}, \mathbf{b}, \mathbf{c}) .$$

Hence

$$P_m(x) = \sum_{\mathbf{b}, \mathbf{c}} R_j(\mathbf{x}, \mathbf{b}, \mathbf{c})$$

which shows that  $P \in PD^1$ . □

We can attempt to generalise the definition of the above vacuous hierarchy by allowing the number of "alternations" to increase with the number of indeterminates.

Let  $t$  be any polynomial. Define  $t-D^0$  to be the class of  $t$ -computable families. For  $i > 0$  let  $t-D^i$  be the class of families that are defined by some family in  $t-D^{i-1}$  in the sense of Definition 3. Finally  $PD^*$  is the class of all families  $P$  such that for some  $t$

$$P = \{P_i \mid P_i = Q_i \text{ for some } Q \in t-D^{(i)}\}.$$

THEOREM 6.  $PD^* = PD^1$

*Proof.* Similar to previous theorem. □

The above two results should be contrasted with the Boolean case where they still hold formally, but are no longer natural. The above definition of the successive levels  $PD^i$  is only natural if each level is a robust closure class. In Boolean algebra, however,  $PD^i$  is not known to be closed under complementation for any  $i \geq 1$ . Analogues of  $PD^i$  and  $PD^*$  where complementation is allowed at each level of alternation are not known to collapse, and are merely finite versions of the Meyer-Stockmeyer hierarchy, and PSPACE respectively [10].

A simple application of Theorem 5 is in recognising such polynomials as  $\#HG$  as being  $p$ -definable. An intriguing open question is whether  $HG$  itself is  $p$ -definable for each  $F$ . If it is not then  $P \neq NP$  (see Proposition 4 in [13]). If it is then the Meyer-Stockmeyer hierarchy and PSPACE can be simulated within  $p$ -definable families of polynomials.

## 6. UNIVERSALITY OF LINEAR PROGRAMMING

Here we consider a Boolean function family  $LP$  that corresponds to a linear programming problem and show that every  $p$ -computable family is the  $p$ -projection of it. Thus for computing discrete functions in polynomial time a package for  $LP$  for each input size is sufficient and no further programming is required. If we fix certain of the arguments of  $LP_i$  according to the particular function and input size being computed, the package becomes a program for the required function. That  $LP$  is itself  $p$ -computable follows from the recent result of Khachian [8].

The reader should note that several tractable problems in combinatorial optimisation are already known to have linear programming formula-