3. Integral representation of a class of second order processes

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where $\| F \| = \| F \| (\mathbf{R} \times \mathbf{R})$. It should be noted, however, that the integrability of (f, g) generally need not imply that of $(|f|, |g|)$, and the MT-integral is *not* an absolutely continuous functional in contrast to the Lebesgue-Stieltjes theory, as already shown by counterexamples in [26] and [27]. Fortunately a certain dominated convergence theorem ([27], Thm. 3.3) is valid and this implies some density properties which can and will be utilized in our treatment below. Also f is termed F-integrable if (f, f) is MT-integrable. Our definition above is somewhat more restrictive than that of [27], but it suffices for this work. For the theory of [27], the space $B(\mathbf{R}, \mathscr{B}, \mathbf{C})$ in (12) and (13) is replaced by $C_{00}(\mathbf{R})$, its subset of continuous functions with compact supports, with the locally convex (inductive limit) topology. Note that, thus far, no special properties of $\mathbf{R}$ were used in the definition of the MT- integral, and the *definition and properties are valid if* $\mathbf{R}$ *is replaced by an arbitrary locally compact space (group in the present context).* This remark will be utilized later on.

With this necessary detour, the second concept can be given as follows:

Definition 2.2. A process $X : \mathbf{R} \to L_0^2(P)$, with $r(\cdot, \cdot)$ as its covariance function, is called *weakly harmonizable* if

$$r(s, t) = I(e^{is(\cdot)}, e^{it(\cdot)}) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda - it\lambda'} F(d\lambda, d\lambda'), \; s, t \in \mathbf{R}, \tag{18}$$

relative to some positive definite bimeasure F of finite semivariation where the right side is the MT-integral.

In particular r is bounded and continuous (by (17) and Thm. 3.2 below). Moreover, if F is of bounded variation, then the MT-integral reduces to the Lebesgue-Stieltjes integral and (18) goes over to (3). The following work shows that the process of the counterexample following Definition 2.1 is weakly harmonizable. The same counterexample also shows that harmonizable processes generally do *not* admit shift operators on them, in that there need not be a continuous linear operator

$$\tau_s : X(t) \mapsto X(t+s) \in L_0^2(P), \qquad t \in \mathbf{R}$$

on $L_0^2(P)$. This is in distinction to certain other nonstationary processes of Karhunen type (cf. [9]).

3. Integral representation of a class of second order processes

In order to introduce and utilize the "V-boundedness" concept of Bochner's, it will be useful to have an integral representation of weakly harmonizable processes. This is done by presenting a comprehensive result for a more general class including the (weakly) harmonizable ones. It is based on a method of Cramér's [3], and the resulting representation yields by specializations both the harmonizable, stationary, Cramér class of [3], as well as the Karhunen class (restated below). This is detailed as follows.

Recall that if $(\Omega_0, \mathscr{A})$ is a measurable space (i.e., $\mathscr{A}$ is a σ-algebra of sets of Ω_0) and $\mathscr{X}$ a Banach space, then a mapping $Z : \mathscr{A} \to \mathscr{X}$ is called a *vector measure* if Z is σ-additive, or

$$Z\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} Z(A_i), \qquad A_i \in \mathscr{A},$$

disjoint, the series converging unconditionally in the norm of $\mathscr{X}$. If $\mathscr{X} = L_0^2(P)$ where (Ω, Σ, P) is a probability space, then a vector measure is sometimes termed a *stochastic measure*. The integration of scalar functions relative to a vector measure Z is needed, and it will be in the sense of Dunford-Schwartz ([8], IV.10). This may be briefly outlined here. If $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, $A_i \in \mathscr{A}$, disjoint, define as usual

$$\int_A f(s)Z(ds) = \sum_{i=1}^{n} a_i Z(A \cap A_i) \in \mathscr{X}, \qquad A \in \mathscr{A}. \tag{19}$$

Now if $g : \Omega_0 \to \mathbf{C}$ is $\mathscr{A}$-measurable, and g_n are $\mathscr{A}$-step functions such that $g_n \to g$ pointwise, one says that g is *D-S integrable* whenever for each $A \in \mathscr{A}$,

$$\Big\{\int_A g_n(s)Z(ds),\, n \geqslant 1\Big\} \subset \mathscr{X}$$

is a Cauchy sequence. Then the limit, denoted g_A, of this sequence is called the integral of g on A, and is denoted as

$$g_A = \int_A g(s)Z(ds) = \lim_{n\to\infty} \int_A g_n(s)Z(ds), \qquad A \in \mathscr{A}. \tag{20}$$

It is a standard (but non-obvious) matter to show that the integral is well-defined, independent of the sequence used, and the mapping $A \mapsto \int_A g(s)Z(ds)$ is σ-additive on $\mathscr{A}$, and $g \mapsto \int_A g(s)Z(ds)$ is linear. Also

$$\Big\| \int_A g(s)Z(ds) \Big\| \leqslant \| g \|_u \, \| Z \| (A), \qquad g \in B(\Omega, \mathscr{A}, \mathbf{C}), \tag{21}$$

where $\| Z \| (\cdot)$ is the semivariation of Z (cf. (7)) which is always finite on the σ-algebra $\mathscr{A}$. [If $\mathscr{A}$ is only a δ-ring and $\Omega_0 \notin \mathscr{A}$, then Z need not have finite semivariation on $\mathscr{A}$.] The dominated convergence theorem is true for the D-S integral. (See [8], IV.10, for proofs and related results. The latter exposition is very readable and nice.)

The general class noted above is the following:

Definition 3.1. A process $X : \mathbf{R} \to L_0^2(P)$, with covariance $r(\cdot, \cdot)$, is said to be *weakly of class* (C) (C for Cramér) if (i) there exists a covariance bimeasure F on $\mathbf{R} \times \mathbf{R}$ of locally bounded semivariation in the sense that

$$F(A, B) = \bar{F}(B, A), \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j F(A_i, A_j) \geqslant 0 , \qquad a_i \in \mathbf{C} .$$

Here $A_i \in \mathscr{B}$, $1 \leqslant i \leqslant n$, bounded, and for each bounded Borel $A \subset \mathbf{R}$, if $\mathscr{B}(A) = \{A \cap B : B \in \mathscr{B}\}$, then

$$\| F \| (A \times A) = \sup \{ | \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{b}_j F(A_i, B_j) | : | a_i | \leqslant 1, | b_j | \leqslant 1 , \\ A_i, B_j \in \mathscr{B}(A), \text{disjoint}\} < \infty ;$$

(ii) there exists an MT-integrable (for F) family $g_t : \mathbf{R} \to \mathbf{C}$ of Borel functions, $t \in \mathbf{R}$, such that $I(|g_s|, |g_s|) < \infty$, $s \in \mathbf{R}$, where I denotes the MT-integral relative to F, in terms of which one has ($g_t(\lambda)$ is also written as $g(t, \lambda)$):

$$r(s, t) = I(g_s, \bar{g}_t) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda) \bar{g}_t(\lambda') F(d\lambda, d\lambda') , \qquad s, t \in \mathbf{R} . \tag{22}$$

Remark. Note that in this definition F can be given by a covariance function ρ as in (3′) since, for $A = [a, b)$ and $B = [c, d)$ one defines $(\Delta^2 F)(A, B)$ as the increment $\rho(b, d) - \rho(a, d) - \rho(b, c) + \rho(a, c)$ and extend it to $\mathscr{B} \times \mathscr{B}$. Also in (22) it is possible that $\| F \| (\mathbf{R} \times \mathbf{R}) = \infty$. If F has finite variation on each compact rectangle of $\mathbf{R}^2$, then F determines a locally bounded complex Radon measure, and the above class reduces to the family defined by Cramér in [3], and called class (C) and analyzed in [35]. If $\| F \| (\mathbf{R} \times \mathbf{R}) < \infty$, then one can take $g_t(\lambda) = g(t, \lambda) = e^{it\lambda}$ so that the weakly harmonizable class is included. Again it may be noted that $\mathbf{R}$ can be replaced by a locally compact space or an abelian group in (22) so that $\mathbf{R}^n$ or the n-torus $\mathbf{T}^n$ is included.

To present the general representation, it is necessary also to note the validity of the D-S integration embodied in (20), (21) when the set functions are defined on arbitrary δ-rings instead of σ-algebras, assumed in [8]. Further our measure $Z : \tilde{\mathscr{B}} \to \mathscr{X}$ has the property that it is Baire regular in the sense that for each $A \in \tilde{\mathscr{B}}$ and $\varepsilon > 0$, there exist a compact $C \in \tilde{\mathscr{B}}$, open $U \in \tilde{\mathscr{B}}$ such that $C \subset A \subset U$ and $\| Z(D) \| < \varepsilon$ for each $D \in \tilde{\mathscr{B}}$, $D \subset U - C$, where $\tilde{\mathscr{B}}$ is the Baire (= Borel here) σ-ring of $\mathbf{R}$. Even if $\mathbf{R}$ is replaced by a general locally compact space S, with $\tilde{\mathscr{B}}$ as its Baire σ-ring and $Z : \tilde{\mathscr{B}} \to \mathscr{X}$ σ-additive, one has Z to be Baire regular having a unique regular extension to the Borel σ-ring $\mathscr{B}$ of S.

Actually Z concentrates on a σ-compact Baire set $S_0 \subset S$. Moreover if Z is weakly regular in that $x^* \circ Z$ is a scalar regular signed measure, $x^* \in \mathscr{X}^*$, then Z is itself regular. (See [21], pp. 262-263 for proofs with only simple modifications of the arguments given in [8], IV.10.) In each case the measure Z has finite semivariation on bounded sets in $\tilde{\mathscr{B}}$ (cf. (7) where $\mathscr{B}$ is replaced by the ring generated by all bounded Baire sets for S). If $\mathscr{B}_0 \subset \mathscr{B}$ is the class of all bounded sets (a set is bounded if it is contained in a compact set), then it is a δ-ring, and the D-S integration of a scalar function relative to $Z : \mathscr{B}_0 \to \mathscr{X}$ holds as noted above. With this understanding the following is the desired general result.

THEOREM 3.2. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a process which is weakly of class* (C) *in the sense of Definition 3.1, relative to a positive definite bimeasure* F *of locally finite semivariation, and a family* $\{g_s, s \in \mathbf{R}\}$ *of Borel functions such that each* $|g_s|$ *is MT-integrable for* F. *Then there exists a stochastic measure* $Z : \mathscr{B}_0 \to L_0^2(\tilde{P})$ *where* $\mathscr{B}_0$ *is the* δ*-ring of bounded Borel sets of* $\mathbf{R}$, *and* $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ *is an enlargement of* (Ω, Σ, P) *so* $L_0^2(\tilde{P}) \supset L_0^2(P)$, *such that*

(i) $E(Z(A) \cdot \overline{Z}(B)) = (Z(A), Z(B)) = F(A, B),\ A, B \in \mathscr{B}_0$,

(ii) $X(t) = \int_{\mathbf{R}} g(t, \lambda) Z(d\lambda),\ t \in \mathbf{R}$, (23)

where the integral is in the D-S *sense for the* δ*-ring* $\mathscr{B}_0$.

Conversely, if $\{X(t), t \in \mathbf{R}\}$ *is a process defined by* (23) *relative to a stochastic measure* $Z : \mathscr{B}_0 \to L_0^2(P)$ *and a Borel family* $\{g_t, t \in \mathbf{R}\}$, D-S *integrable for* Z *and* $\mathscr{B}_0$, *then it is weakly of class* (C) *relative to* F *defined by*

$$F(A, B) = E(Z(A) \cdot \overline{Z}(B)), \qquad A, B \in \mathscr{B}_0,$$

and each $|g_t|, t \in \mathbf{R}$, *is MT-integrable for* F. *Moreover, if*

$$\mathscr{H}_X = \overline{sp}\{X(t), t \in \mathbf{R}\}$$

and

$$\mathscr{H}_Z = \overline{sp}\{Z(A), A \in \mathscr{B}_0\}$$

in $L_0^2(P)$, *then* $\mathscr{H}_X = \mathscr{H}_Z$ *when and only when the* $\{g_t, t \in \mathbf{R}\}$ *has the property that*

$$\int_{\mathbf{R}} \int_{\mathbf{R}} f(\lambda) \bar{g}_t(\lambda') F(d\lambda, d\lambda') = 0, \qquad \textit{all } t \in \mathbf{R},$$

implies $\int_{\mathbf{R}} \int_{\mathbf{R}} f(\lambda) \bar{f}(\lambda') F(d\lambda, d\lambda') = 0$ *both being* MT*-integrals.*

Proof: The basic layout is that of [3]. The integrals used there will have to be replaced by the D-S and MT-integrals appropriately. Since the changes are not immediately obvious, the essential details are spelled out so that in subsequent discussions, such arguments can be compressed.

For the direct part, let the process be weakly of class (C). Then its covariance r admits a representation (with the MT-integration) as:

$$r(s, t) = E(X(s)\bar{X}(t)) = \int_{\mathbf{R}}\int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda'). \tag{24}$$

Since F is a positive definite bimeasure, if

$$L_F^2 = \{f: \int_{\mathbf{R}}\int_{\mathbf{R}} f(\lambda)\bar{f}(\lambda')F(d\lambda, d\lambda') = (f, f)_F < \infty\,,\ f \text{ is MT-integrable for } F\},$$

and since $I_F(f, f) = (f, f)_F \geqslant 0$, the earlier discussion implies $\{L_F^2, (\cdot, \cdot)_F\}$ is a semi-inner product space, and $g_t \in L_F^2$, $t \in \mathbf{R}$. Let $T: L_F^2 \to \mathscr{H}_X$ be defined by $T: g_s \mapsto X(s)$, extending it linearly. Then (24) implies

$$(Tg_s, Tg_t)_{\mathscr{H}_X} = (g_s, g_t)_F, \qquad s, t \in \mathbf{R}. \tag{25}$$

Thus T is an isometric mapping of $\Lambda_F^2 = sp\{g_t, t \in \mathbf{R}\} \subset L_F^2$ onto $\mathscr{H}_X$ where $\mathscr{H}_X$ is the space given in the statement of the theorem.

Suppose first that Λ_F^2 is dense in L_F^2. By ([27], Thm. 11.1) every Borel function with $I^*(|f|, |f|) < \infty$ is in L_F^2, so that, in particular $\chi_A \in L_F^2$ for each $A \in \mathscr{B}_0$ since F is locally of finite semivariation. By the density of Λ_F^2 in L_F^2 and the isometry, there is a $Z_A \in \mathscr{H}_X$ such that $T\chi_A = Z_A$. If $A, B \in \mathscr{B}_0$, then

$$E(Z_A \cdot \bar{Z}_B) = (T\chi_A, T\chi_B)_{\mathscr{H}_X} = (\chi_A, \chi_B)_F = F(A, B),$$

and if $A \cap B = \emptyset$ also holds, then

$$E(|Z_{A\cup B} - Z_A - Z_B|^2) = (\chi_{A\cup B} - \chi_A - \chi_B, \chi_{A\cup B} - \chi_A - \chi_B)_F = 0$$

since F is additive in both components. Thus $Z_{(\cdot)}: \mathscr{B}_0 \to \mathscr{H}_X \subset L_0^2(P)$ is additive. If $\{A_n\}_1^\infty \subset \mathscr{B}_0$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{B}_0$, then

$$E(|Z_A - \sum_{i=1}^{n} Z_{A_i}|^2) = E(|Z_{\bigcup_{i=1}^{n} A_i} + Z_{\bigcup_{i>n} A_i} - \sum_{i=1}^{n} Z_{A_i}|^2)$$

$$= E(|Z_{\bigcup_{i>n} A_i}|^2) = F(\bigcup_{i>n} A_i, \bigcup_{i>n} A_i) \to 0$$

as $n \to \infty$, since F is continuous at $\emptyset$ from above (cf. discussion after (7)). This Z is σ-additive on $\mathscr{B}_0$ and hence is a stochastic measure of finite semivariation on each compact set there. Clearly $\mathscr{H}_Z \subset \mathscr{H}_X$. Since $\{g_t, t \in \mathbf{R}\}$ is dense in L_F^2, $\chi_A \in L_F^2$, and each g_t is assumed MT-integrable for F, there is a sequence $\tilde{g}_n = \sum_{i=1}^{n} a_i g_{t_i} \to \chi_A$ in L_F^2 so that $(\tilde{g}_n - \chi_A, \tilde{g}_n - \chi_A)_F \to 0$. Hence by the isometry $E(|\sum_{i=1}^{n} a_i X(t_i) - Z_A|^2) \to 0$, as $n \to \infty$. It now follows easily that $\{Z_A, A \in \mathscr{B}_0\}$ is dense in $\mathscr{H}_X$. Thus $\mathscr{H}_X = \mathscr{H}_Z$, and each element in $\mathscr{H}_Z$ corresponds uniquely to an element of $\bar{L}_F^2$, the completion of L_F^2 and where elements $h \in \bar{L}_F^2$ with $(h, h)_F = 0$ and 0 are identified. Let $Y(t)$ be defined as:

$$Y(t) = \int_{\mathbf{R}} g_t(\lambda) Z(d\lambda) \qquad \in \mathscr{H}_Z = \mathscr{H}_X. \tag{26}$$

Here the right side is the D-S integral on the δ-ring $\mathscr{B}_0$, which can be defined by a slight modification of the work of ([8], IV.10), as noted in [21]. Thus,

$$\begin{aligned}(Y(s), Y(t)) &= \Big(\int_{\mathbf{R}} g_s(\lambda) Z(d\lambda), \int_{\mathbf{R}} g_t(\lambda') Z(d\lambda')\Big) \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} g_s(\lambda) \bar{g}_t(\lambda') F(d\lambda, d\lambda')\end{aligned}$$

which holds if g_s is a $\mathscr{B}_0$-measurable step function and then the general case follows by ([27], Thm. 3.3 or [46], p. 126), since $|g_s|$ is MT-integrable in our sense. Now by definition (l·i·m denoting $L^2(P)$-mean):

$$\begin{aligned}Z(A) &= T(\chi_A) = T(\lim_n \tilde{g}_n), \quad \text{where} \quad \tilde{g}_n \to \chi_A \quad \text{in} \quad L_F^2 \\ &= \underset{n}{\mathrm{l\cdot i\cdot m}}\, T(\tilde{g}_n) = \underset{n}{\mathrm{l\cdot i\cdot m}} \sum_{i=1}^{n} a_i\, T(g_{t_i}) \\ &= \underset{n}{\mathrm{l\cdot i\cdot m}} \sum_{i=1}^{n} a_i\, X(t_i) = \underset{n}{\mathrm{l\cdot i\cdot m}}\, \tilde{X}_n \quad \text{(say)}.\end{aligned}$$

The $L^2(P)$-limits imply

$$\begin{aligned}E(X(s)\bar{Z}(A)) &= \lim_n E(X(s)\overline{\tilde{X}_n}) \\ &= \lim_n \sum_{i=1}^{n} a_i\, E(X(s)\bar{X}(t_i)) = \lim_n \sum_{i=1}^{n} a_i\, r(s, t_i) \\ &= \lim_n \sum_{i=1}^{n} a_i \int_{\mathbf{R}}\int_{\mathbf{R}} g_s(\lambda)\bar{g}_{t_i}(\lambda') F(d\lambda, d\lambda') \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} g_s(\lambda)\chi_A(\lambda') F(d\lambda, d\lambda').\end{aligned}$$

By isometry, if $\zeta_n = \sum_{j=1}^{n} b_j \cdot Z(A_j)$, one gets $\tilde{h}_n \leftrightarrow \zeta_n$ where $\tilde{h}_n = \sum_{j=1}^{n} b_j \chi_{A_j} \in L_F^2$,

$$E(X(s)\bar{\zeta}_n) = \int_{\mathbf{R}}\int_{\mathbf{R}} g_s(\lambda)\bar{\tilde{g}}_n(\lambda')F(d\lambda, d\lambda') .$$

So again by the MT-integrability of $g_s(\cdot)$, the preceding result yields

$$E(X(s)\bar{Y}(t)) = \int_{\mathbf{R}}\int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda') .$$

It follows from this that

$$E(|X(s) - Y(s)|^2) = E(|X(s)|^2) + E(|Y(s)|^2) - E(X(s)\overline{Y(s)}) - E(Y(s)\overline{X(s)}) = 0 .$$

Hence $X(s) = Y(s)$ *a.e.*, $s \in \mathbf{R}$. So (26) implies (23) in the event that Λ_F^2 is dense in L_F^2.

For the general case, where $\tilde{\Lambda}_F^2 = \bar{L}_F^2 \ominus \Lambda_F^2$ is nontrivial and where the "bar" again denotes completion, let $\{h_t, t \in \tilde{R}\}$ be a basis of $\tilde{\Lambda}_F^2$. If $\tilde{\tilde{R}} = \mathbf{R} \dotplus \tilde{R}$ is a disjoint sum to give a new index set, let $\tilde{g}_s = g_s$ for $s \in \mathbf{R}$, and $= h_s$ for $s \in \tilde{R}$, then $\{\tilde{g}_s, s \in \tilde{\tilde{R}}\}$ is dense in $\bar{L}_F^2$. So by the preceding case, on extending T to τ from $L_F^2 \to L_0^2(\tilde{P})$, where $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ is possibly an enlargement of (Ω, Σ, P) by adjunction (cf., e.g., [36], p. 82), with $\tau\chi_A = Z_A \in L_0^2(\tilde{P})$, one has

$$\tilde{Y}(s) = \int_{\mathbf{R}} \tilde{g}_s(\lambda)Z(d\lambda) \qquad \in L_0^2(P) . \tag{27}$$

Observe that all $\tilde{g}_s$ are Borel and MT-integrable in this procedure. Hence, as before, $\tilde{Y}(s) = X(s)$ for $s \in \mathbf{R}$, and (23) holds again. In this case $\mathscr{H}_Z \supset \mathscr{H}_X$, and the inclusion is proper.

Conversely, let $\{X(t), t \in \mathbf{R}\}$ be a process defined by (23). Let $F(A, B) = (Z(A), Z(B))$ and $g_n = \sum_{i=1}^{n} a_i\chi_{A_i}$, A_i, A, B in $\mathscr{B}_0$. Then for the D-S integral (23) one has

$$\begin{aligned} \| F \|(A, A) &= \sup \{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{a}_j F(A_i, A_j) : A_i \in \mathscr{B}(A), | a_i | \leqslant 1\} \\ &= \sup \{\| \sum_{i=1}^{n} a_i Z(A_i) \|_2^2 : | a_i | \leqslant 1, A_i \in \mathscr{B}(A)\} \\ &\leqslant \| Z \|^2(A) < \infty, A \in \mathscr{B}_0 . \end{aligned}$$

Thus if $X_{g_n} = \int_{\mathbf{R}} g_n(\lambda)Z(d\lambda)$, one has with h_n another such step function,

$$E(X_{g_n}\bar{X}_{h_n}) = \int_{\mathbf{R}}\int_{\mathbf{R}} g_n(\lambda)\bar{h}_n(\lambda')F(d\lambda, d\lambda') . \tag{28}$$

Now given $g_s \in L_F^2$ which is MT-integrable in our (restricted) sense (this is analogous to a definition of [46]) and for which (23) holds, the g_s can be approximated by suitable Borel step functions $\{g_n\}_1^\infty \subset L_F^2$ such that $g_n \to g_s$ pointwise $|g_n| \leqslant |g_s|$ and similarly with $\tilde{g}_n \to g_t$ such that

$$I(g_n, \tilde{g}_n) \to I(g_s, g_t), I(|g_s|, |g_s|) < \infty .$$

Applying this to (28), one obtains

$$\begin{aligned}\int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda') &= \lim_n \int_{\mathbf{R}} \int_{\mathbf{R}} g_n(\lambda)\bar{\tilde{g}}_n(\lambda)F(d\lambda, d\lambda') \\ &= \lim_n (X_{g_n}, X_{\tilde{g}_n}) \\ &= \lim_n \left(\int_{\mathbf{R}} g_n(\lambda)Z(d\lambda), \int_{\mathbf{R}} \tilde{g}_n(\lambda')Z(d\lambda')\right) \\ &= \left(\int_{\mathbf{R}} g_s(\lambda)Z(d\lambda), \int_{\mathbf{R}} g_t(\lambda)Z(d\lambda)\right),\end{aligned}$$

since for the D-S integral the dominated convergence holds,

$$= (X(s), X(t)) = r(s, t) . \tag{29}$$

This shows $\{X(t), t \in \mathbf{R}\}$ is of weakly class (C).

Regarding the last assertion, it is evident that $\{g_s, s \in \mathbf{R}\}$ is a basis in L_F^2 iff $I(f, g_t) = 0, t \in \mathbf{R}$ implies $I(f, f) = 0$. This is clearly necessary and sufficient for $\mathscr{H}_Z = \mathscr{H}_X$ since otherwise, (with possibly an enlargement of the underlying probability space) $\mathscr{H}_Z \underset{\neq}{\supset} \mathscr{H}_X$ and $\mathscr{H}_Z = \mathscr{H}_{\tilde{Y}}$ in the notation of (27). Thus the proof is complete.

Remarks. 1. If F is of locally finite variation, then it defines a locally finite (i.e., finite on compact sets) complex Borel (= Radon) measure in the plane $\mathbf{R}^2$, and then the MT-integrals for F reduce to the Lebesgue-Stieltjes integrals. Thus $I(g_s, g_s) < \infty$ is equivalent to the classical theory, and the above result specializes to Cramér's theorem of [3]. However, for the general case of bimeasures (as here), the MT-theory (or a form of it) appears essential.

2. The above theorem is true if $\mathbf{R}$ is replaced by a *locally compact space*, since no special property of $\mathbf{R}$ is used. Only the concept of boundedness is needed.

When $\| F \|(\mathbf{R} \times \mathbf{R}) < \infty$, so that F is of finite semivariation on $\mathbf{R}^2$, then by ([27], Thm. 11.1) each bounded Borel function g is MT-integrable for F. Taking $g_t(\lambda) = e^{it\lambda}$ in the above theorem, one deduces from this result the important representation given by Rozanov ([40], p. 279). The last statement is not too hard to establish. [A separate proof of it is also found in ([29], p. 36).]

THEOREM 3.3. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a process such that* $\| X(t) \|_2 \leqslant M < \infty, t \in \mathbf{R}$, *and be weakly continuous. Then the process is weakly harmonizable relative to some covariance bimeasure* F *of finite semivariation* (cf. *Definition* 2.2) *iff there is a stochastic measure* $Z : \mathscr{B} \to L_0^2(P)$ *such that for each* A, B *in* $\mathscr{B}$, $F(A, B) = (Z(A), Z(B))$ *and*

$$X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \qquad t \in \mathbf{R}, \tag{30}$$

the right side symbol being the D-S integral and $\| Z \|(\mathbf{R}) < \infty$. *Moreover,* X *is strongly harmonizable iff the covariance bimeasure* F *of* Z *in* (30) *is of bounded variation in* $\mathbf{R}^2$ (cf. *Definition* 2.1). *In either case the harmonizable process* X *is uniformly continuous, and is represented as in* (30).

Suppose that in the representation (23) the Z-process is *orthogonally scattered* implying $(Z(A), Z(B)) = 0$ whenever $A \cap B = \emptyset$. Then

$$F(A, B) = (Z(A), Z(B)) = \tilde{F}(A \cap B),$$

where F is the covariance bimeasure and $\tilde{F}$ is a positive locally finite measure on $\mathcal{B}$ so that it is σ-finite there. Then

$$r(s, t) = E(X_s \bar{X}_t) = \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda)\tilde{F}(d\lambda). \tag{31}$$

A process whose covariance function $\mathbf{R}$ satisfies this condition is termed a *Karhunen process.* Moreover, if $\tilde{F}$ is a finite measure and $g_s(\lambda) = e^{is\lambda}$, the resulting one is the classical (Khintchine) stationary process. In both these cases there are no weak type extensions.

Let us introduce a further generalization of the (weak) Cramér class to illuminate the above Definition 3.1, and for a future analysis. Let (Ω, Σ, μ) be a measure space and $M(\mu)$ be the space of scalar μ-measurable functions on Ω. Let $N(\cdot) : M(\mu) \to \mathbf{R}^+$ be a *function norm* in that for f, f_n in $M(\mu)$, (i) $N(f) = N(|f|) \geqslant 0$, (ii) $0 \leqslant f_n \uparrow \Rightarrow N(f_n) \uparrow$, (iii) $N(af) = |a| N(f)$, $a \in \mathbf{C}$ and (iv) $N(f+g) \leqslant N(f) + N(g)$. The functional N has the weak Fatou property if

$$0 \leqslant f_n \uparrow f, \lim_n N(f_n) < \infty \Rightarrow N(f) < \infty,$$

and has the Fatou property if instead $N(f_n) \uparrow N(f)$ $(\leqslant \infty)$. The associate norm N' of N is defined by:

$$N'(f) = \sup \left\{ \left| \int_\Omega (fg)(\omega)\mu(d\omega) \right| : N(g) \leqslant 1 \right\}. \tag{32}$$

One sees that N' is a function norm with the Fatou property. If

$$N(\cdot) = \| \cdot \|_p, \ 1 \leqslant p \leqslant \infty,$$

then

$$N'(\cdot) = \| \cdot \|_q, \ p^{-1} + q^{-1} = 1.$$

The general concept alluded to above is as follows:

Definition 3.4. (a) If $r : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ is a covariance function, it is said to be of *class*$_N$ (C) relative to a function norm N, if there is a covariance bimeasure

$F : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ of locally finite N-variation (let N' be the associate norm of N), and there exists a family $\{g_t, t \in \mathbf{R}\}$ of Borel functions which are MT-integrable relative to F, such that

$$r(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda'), \; s, t \in \mathbf{R}\,, \tag{33}$$

and where locally finite N-variation is meant the following:

$$\infty > \| F \|_N(A \times A) = \sup \{ | I(f, g) | : N'(f) \leqslant 1, N'(g) \leqslant 1 \}\,. \tag{34}$$

Here f, g are Borel step functions, with $\mathrm{supp}(f) \subset A$, $\mathrm{supp}(g) \subset A$, $A \in \mathscr{B}_0$, the δ-ring of bounded Borel sets of $\mathbf{R}$.

(b) A process $X : \mathbf{R} \to L_0^2(P)$ is of $\mathrm{class}_N(\mathrm{C})$ if its covariance function r is of class_N (C) so that it is representable as (33).

It is clear that if $N(\cdot) = \| \cdot \|_1$ so that $N'(\cdot) = \| \cdot \|_\infty$, the N-variation is simply the 1-semivariation of Definition 3.1 and that

$$\| F \|_N = \| F \|_1 \,(= \| F \|)\,.$$

Remark. Without further restrictions, class_N (C) need not contain the weak or strong harmonizable processes. However if N is restricted so that, letting

$$L^N(P) = \{f \in M(P) : N(f) < \infty\},\; L^\infty(P) \subset L^N(P) \subset L^1(P)\,,$$

where $\mu = P$ is a probability, then every class_N (C) will contain both the weak and strong harmonizable families, as an easy computation shows. If $N(\cdot) = \| \cdot \|_1$, then class_1 (C) is the class which corresponds to the covariance bimeasure of *finite semivariation*. This includes the classical Loève and Rozanov definitions. Again this definition holds, with only a notational change, if $\mathbf{R}$ is replaced by a locally compact group G. A brief discussion on some analysis of these classes which extend the present work is included at the end of the paper.

4. V-BOUNDEDNESS, WEAK AND STRONG HARMONIZABILITY

The definition of weak harmonizability is of interest only when an effective characterization of it is found and when its relations with strong harmonizability are made concrete. These points will be clarified and answered here. Now Theorem 3.3 shows that a weakly harmonizable process is the Fourier transform of a stochastic measure and this leads to a fundamental concept called V-boundedness ('V' for "variation"), introduced much earlier by Bochner [2], which is valid in a more general context. This notion plays a central role in the theory and applications of weakly harmonizable processes (and fields) which are