5. Domination problem for harmonizable fields

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$$0 \leqslant \int_{\mathbf{R}} \int_{\mathbf{R}} f(s)\overline{f(t)} F_n(ds, dt) \leqslant \int_{\mathbf{R}} \int_{\mathbf{R}} |f(s)\overline{f(t)}| \, |F_n|(ds, dt)$$

$$\leqslant \frac{1}{2} [\int_{\mathbf{R}} \int_{\mathbf{R}} |f(s)|^2 \, |F_n|(ds, dt) + \int_{\mathbf{R}} \int_{\mathbf{R}} |f(t)|^2 \, |F_n|(ds, dt)],$$

since $|ab| \leqslant (|a|^2 + |b|^2)/2$,

$$= \int_{\mathbf{R}} |f(s)|^2 \beta_n(ds), \qquad \text{by (54)}. \tag{55}$$

This and (53) yield

$$\alpha_0^f = \| \int_{\mathbf{R}} f(\lambda) Z(d\lambda) \|_2^2 = \lim_n \int_{\mathbf{R}} \int_{\mathbf{R}} f(\lambda)\overline{f}(\lambda') F_n(d\lambda, d\lambda')$$

$$\leqslant \liminf_n \int_{\mathbf{R}} |f(\lambda)|^2 \beta_n(d\lambda), \qquad f \in C_0(\mathbf{R}). \tag{56}$$

This completes the proof.

Remark. For a deeper analysis of the structure of these processes, it is desirable to replace the sequence $\{\beta_n, n \geqslant 1\}$ by a single Borel measure. This is nontrivial. In the next section for a more general version, including harmonizable fields, such a result will be obtained.

5. Domination problem for harmonizable fields

The work of the preceding section indicates that the weakly harmonizable processes are included in the class of functions which are Fourier transformations of vector measures into Banach spaces. A characterization of such functions, based on the V-boundedness concept of [2], has been obtained first in [33]. For probabilistic applications (e.g., filtering theory) the domination problem, generalizing Theorem 4.5, should be solved. The following result illuminates the nature of the general problem under consideration.

THEOREM 5.1. *Let* (Ω, Σ) *be a measurable space,* $\mathscr{X}$ *a Banach space and* $\nu : \Sigma \to \mathscr{X}$ *be a vector measure. Then there exists a (finite) measure* $\mu : \Sigma \to \mathbf{R}^+$, *a continuous convex function* $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\frac{\varphi(x)}{x} \nearrow \infty$ *as* $x \nearrow \infty$, $\varphi(0) = 0$, *and* ν *has* φ*-semivariation finite relative to* μ *in the sense that*

$$\| \nu \|_{\varphi}(\Omega) = \sup \{ \| \int_{\Omega} f(\omega) \nu(d\omega) \|_{\mathscr{X}} : \| f \|_{\psi, \mu} \leqslant 1 \} < \infty, \tag{57}$$

where $\| f \|_{\psi, \mu} = \inf \{ \alpha > 0 : \int_{\Omega} \psi \left(\frac{|f(\omega)|}{\alpha} \right) \mu(d\omega) \leqslant 1 \} < \infty$, *and the* *integral relative to* ν *in (57), is in the Dunford-Schwartz sense. Here* $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ *is a convex function given by* $\psi(x) = \sup\{|x| y - \varphi(y) : y \geq 0\}$.

The proof of this result depends on some results of ([8], IV.10) and elementary properties of Orlicz spaces (cf. [47], p. 173). It will be omitted here, since the details are given in [38]. This is only motivational for what follows.

Note that (57) is a desired generalization of (51) if $\{\beta_n, n \geq 1\}$ is replaced by μ and $\|\cdot\|_2$ is replaced by $\|\cdot\|_\varphi$. However φ may grow faster than a polynomial. What is useful here is a φ with $\varphi(x) = |x|^p$, $1 \leq p \leq 2$. This can be proved for a special class of spaces $\mathscr{X}$, which is sufficient for our study of harmonizable fields.

It will be convenient to introduce a definition and to state a result (essentially) of Grothendieck and Pietsch, for the work below.

Definition 5.2. Let $\mathscr{X}, \mathscr{Y}$ be a pair of Banach spaces and, as usual, $B(\mathscr{X}, \mathscr{Y})$ be the space of bounded linear operators on $\mathscr{X}$ into $\mathscr{Y}$. If $1 \leq p \leq \infty$, $T \in B(\mathscr{X}, \mathscr{Y})$, then T is called *p-absolutely summing* if $\alpha_p(T) < \infty$, where

$$\alpha_p(T) = \inf\{c > 0 : [\sum_{i=1}^{n} \| Tx_i \|^p]^{\frac{1}{p}} \leq c \sup_{\|x^*\| \leq 1} (\sum_{i=1}^{n} | x^*(x_i) |^p)^{\frac{1}{p}}, x_i \in \mathscr{X}, \\ 1 \leq i \leq n, n \geq 1\}, \tag{58}$$

with $x^* \in \mathscr{X}^*$, the adjoint space of $\mathscr{X}$.

The following result, which is alluded to above, with a short proof may be found in [22] together with some extensions and applications.

PROPOSITION 5.3. *Let* $T \in B(\mathscr{X}, \mathscr{Y})$ *be p-absolutely summing,* $1 \leq p < \infty$. *Let* K^* *be the weak-star closure of the set of extreme points of the unit ball* U^* *of* $\mathscr{X}^*$. *Then there is a regular Borel probability measure* μ *on the compact space* K^* *such that*

$$\| Tx \|_{\mathscr{Y}} \leq \alpha_p(T) [\textstyle\int_{K^*} | x^*(x) |^p \mu(dx^*)]^{1/p}, \qquad x \in \mathscr{X}. \tag{59}$$

Conversely (and this is simple), if T *satisfies (59) for some* μ *on* K^* *with a constant* γ_0, *then* T *is p-absolutely summing and* $\alpha_p(T) \leq \gamma_0$. *Further any p-absolutely summing operator is weakly compact.*

Let us specialize this result in the case that $\mathscr{X} = C_r(S)[C(S)]$, the space of real [complex] continuous functions on a compact set S. Let K be the set of all extreme points of the unit ball U^* of $(C_r(S))^*$ and $q : S \to (C_r(S))^*$ be the mapping defined by $q(s) = l_s$ with $l_s(f) = f(s)$, $f \in C_r(S)$ so that l_s is the evaluation functional, $\| l_s \| = 1$, and $l_s \in K$, $s \in S$. Some other known results needed from Linear Analysis, in the form used here, are as follows. (For details, see [4], Sec. V.3; [8], p. 441). In this case the spaces S and $q(S)$ are homeomorphic and $q(S)$ is closed since S is compact. By Mil'man's theorem U^* is the weak-star closed convex hull of $q(S) \cup (-q(S))$, and (by the compactness of S) the latter is equal to the extreme point-set of U^* and is closed. Further these are of the form αl_s, $s \in S$, and $|\alpha| = 1$ (cf. [8], V.8.6). Consequently (59) becomes

$$\begin{aligned} \| Tf \|^p &\leqslant (\alpha_p(T))^p \cdot \int_{q(S)\cup(-q(S))} | l_s(f) |^p \mu(dl_s), \qquad f \in C_r(S) \\ &\leqslant 2(\alpha_p(T))^p \cdot \int_{q(S)} | l_s(f) |^p \mu(dl_s), \\ &= 2(\alpha_p(T))^p \cdot \int_S | f(s) |^p \mu(ds), \end{aligned}$$

if S and $q(S)$ are (as they can be) identified. For the complex case, $C(S) = C_r(S) + iC_r(S)$, and so the same holds if the constants are doubled. Thus

$$\| Tf \| \leqslant C_p[\int_S | f(s) |^p \mu(ds)]^{\frac{1}{p}} = C_p \| f \|_{p,\mu}, \qquad f \in C(S), \tag{60}$$

where $C_p^p = 4[\alpha_p(T)]^p$. This form of (59) will be utilized below.

Definition 5.4. Let $\mathscr{X}$ be a Banach space, $1 \leqslant p \leqslant \infty$ and $1 \leqslant \lambda < \infty$. Then $\mathscr{X}$ is called *an* $\mathscr{L}_{p,\lambda}$*-space* if for each n-dimensional space $E \subset \mathscr{X}$, $1 \leqslant n < \infty$. there is a finite dimensional $F \subset \mathscr{X}$, $E \subset F$, such that $d(F, l_p^n) \leqslant \lambda$ where l_p^n is the n-dimensional sequence space with p-th power norm and where

$$d(E_1, E_2) = \inf \{\| T \| \, \| T^{-1} \| : T \in B(E_1, E_2)\}$$

for any pair of normed linear spaces E_1, E_2. A Banach space $\mathscr{X}$ is *an* $\mathscr{L}_p$*-space* if it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geqslant 1$.

It is known (and easy to verify) that each $L^p(\mu)$, $p \geqslant 1$, is an $\mathscr{L}_{p,\lambda}$-space for every $\lambda > 1$, and $C(S)$ [indeed each abstract (M)-space] is an $\mathscr{L}_{\infty,\lambda}$-space for every $\lambda > 1$. The class of $\mathscr{L}_2$-spaces coincides with the class of Banach spaces isomorphic to a Hilbert space. For proofs and more on these ideas the reader is referred to the article of Lindenstrauss and Pelczyński [22].

With this set up the following general result can be established at this time on the domination problem for vector measures.

THEOREM 5.5. *Let S be a locally compact space and $C_0(S)$ be the Banach space of continuous scalar functions on S vanishing at "∞". If $\mathscr{Y}$ is an $\mathscr{L}_p$-space $1 \leqslant p \leqslant 2$, and $T \in B(C_0(S), \mathscr{Y})$, then there exist a finite positive Borel measure μ on S, and a vector measure Z on S into $\mathscr{Y}$, such that*

$$\| \int_S f(s) Z(ds) \|_{\mathscr{Y}} = \| Tf \|_{\mathscr{Y}} \leqslant \| f \|_{2,\mu}, \qquad f \in C_0(S). \tag{61}$$

Proof. Since $\mathscr{X} = C_0(S)$ is an abstract (M)-space, it is an $\mathscr{L}_\infty$-space by the preceding remarks. But $\mathscr{Y}$ is an $\mathscr{L}_p$-space $1 \leqslant p \leqslant 2$, and so $T \in B(\mathscr{X}, \mathscr{Y})$ is 2-absolutely summing by ([22], Thm. 4.3), and therefore (cf. Prop. 5.3 above) it is also weakly compact. By the argument presented for (37), (38) above, one can use the theorem ([8], VI.7.3) even when S is locally compact (and noncompact) to conclude that there is a vector measure Z on the Borel σ-ring of S into $\mathscr{Y}$ such that

$$Tf = \int_S f(s)Z(ds), \qquad \text{(D-S integral)}.$$

Using the argument of (37), if $\tilde{S}$ is the one point compactification of S, and $\tilde{T} \in B(C(\tilde{S}), \mathscr{Y})$ is the norm preserving extension, then $\tilde{T}$ is 2-absolutely summing (since $C(\tilde{S})$ is an abstract (M)-space), and weakly compact. So by (60) there exists a finite Borel measure $\tilde{\mu}$ on $\tilde{S}$ such that

$$\| \tilde{T}f \|_{\mathscr{Y}} \leqslant c_p \| f \|_{2,\tilde{\mu}}, \qquad f \in C(\tilde{S}).$$

Letting $\bar{\mu} = c_p^p \tilde{\mu}$, one has $\| \tilde{T}f \| \leqslant \| f \|_{2,\bar{\mu}}$, $f \in C(\tilde{S})$. So (61) holds on $\tilde{S}$. Let $\mu(\cdot) = \bar{\mu}(S \cap \cdot)$ so that μ is a finite Borel measure on S. If now one restricts to $C_0(S)$ identified as a subset of $C(\tilde{S})$, so that $T = \tilde{T} \mid C_0(S)$, it follows from the preceding analysis that $\| Tf \|_{\mathscr{Y}} \leqslant \| f \|_{2,\mu}$ for all $f \in C_0(S)$. Since the integral representation of T is evidently true, this establishes (61), and completes the proof of the theorem.

If $\mathscr{Y}$ is a Hilbert space, it is an $\mathscr{L}_2$-space so that the above theorem considerably strengthens Theorem 4.5, since the sequence there is now replaceable by a single measure.

The following statement is actually a consequence of the above result, and it will be invoked in the last section.

PROPOSITION 5.6. *Let* (Ω, Σ) *be any measurable space, and* $\mathscr{X} = B(\Omega, \Sigma)$ *be the Banach space (under uniform norm) of scalar measurable functions. If* $\mathscr{Y}$ *is an* $\mathscr{L}_p$*-space,* $1 \leqslant p \leqslant 2$, *as above,* $T \in B(\mathscr{X}, \mathscr{Y})$ *is such that for each* $f_n \in \mathscr{X}$, $f_n \to f$ *pointwise boundedly implies* $\| Tf_n \| \to \| Tf \|$, *then there exist* σ*-additive functions* $Z: \Sigma \to \mathscr{Y}$, $\mu: \Sigma \to \mathbf{R}^+$, *such that*

$$\left\| \int_\Omega f(\omega)Z(d\omega) \right\|_{\mathscr{Y}} = \| Tf \|_{\mathscr{Y}} \leqslant \| f \|_{2,\mu}, \qquad f \in \mathscr{X}. \tag{62}$$

The proof uses the fact that $B(\Omega, \Sigma)$ is isometrically isomorphic to $C(S)$, for a compact (extremelly disconnected) Hausdorff space (cf. [8], IV.6.18), and reduces to the preceding result. The computations, using the standard Carathéodory measure theory, will be omitted here. The details, however, may be found in [38].

Remark. The preceding results show that the domination problem for vector measures in L^p-spaces, $1 \leqslant p \leqslant 2$, is solved and hence also for harmonizable fields since only the $\mathscr{L}_2$-type spaces are involved in the latter. But, for $p > 2$, such a satisfactory solution of the problem is not available.