7. Characterizations of weak harmonizability

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.07.2024**

7. Characterizations of weak harmonizability

In this section a different type of characterization, based on the V-boundedness concept crucially, of weak harmonizability as well as a comprehensive statement embodying all the other equivalences of this concept are given. The comparison will illuminate the structure of this general class of processes. However, it is interesting and useful to obtain a characterization of V-boundedness for a general Banach space, and then specialize the result for the harmonizable case.

In this context let us say that $X : G \to \mathscr{X}$, a Banach space, is a *generalized (or vector) Fourier transform* if G is an LCA group, and if there is a vector measure $\nu : \mathscr{B}(\hat{G}) \to \mathscr{X}$ such that $X(g) = \int_{\hat{G}} \langle g, s\rangle \, \nu(ds), g \in G$. In [33], Phillips has extended the fundamental scalar result of Bochner's V-boundedness to certain Banach spaces with $G = \mathbf{R}$. Later but apparently independently, the LCA group case was given by Kluvanek in ([21], p. 269). In the present terminology this can be stated as follows:

Proposition 7.1. *Let G be an LCA group and $\mathscr{X}$ a Banach space. Then a mapping $X : G \to \mathscr{X}$ is a generalized Fourier transform of a regular vector measure $\nu : \mathscr{B}(\hat{G}) \to \mathscr{X}$ (i.e., for given $\varepsilon > 0$ and $E \in \mathscr{B}(\hat{G})$, there exist an open set O and a compact set C with $O \supset E \supset C$ such that for each $F \subset O - C$, $F \in \mathscr{B}(\hat{G})$ one has $\| \nu(F) \| < \varepsilon$) iff X is weakly continuous and V-bounded (in the sense of Definition 4.1).*

On the other hand, when $\mathscr{X} = \mathbf{C}$, a different kind of characterization was given by Helson [12]. A vector extension of this is used for the weak harmonizability problem, and will be presented here. Let $L^k(G)$ be the Lebesgue space, $k \geqslant 1$, on G relative to a Haar measure, denoted dg. Similarly $L^k(\hat{G})$ is defined on the dual group $\hat{G}$, and $L^k_{\mathscr{X}}(\hat{G})$ for $\mathscr{X}$-valued function space. Let

$$\hat{L}^1(G) = \{\hat{f} : \hat{f}(t) = \int_G \langle t, s\rangle f(s)ds, \qquad f \in L^1(G)\} \subset C_0(\hat{G}),$$

a similar definition for $\hat{L}^1_{\mathscr{X}}(G)$, the integrals in the latter being in the sense of Bochner, and $\hat{\mathscr{M}}_{\mathscr{X}}(G)$ $(\supset \hat{L}^1_{\mathscr{X}}(G))$, $\mathscr{M}_{\mathscr{X}}(G)$ being the space of vector measures on G into $\mathscr{X}$ with semivariation norm.

The following result contains the desired extension:

Theorem 7.2. *Let G be an LCA group, $\mathscr{X}$ a reflexive separable Banach space, and $X : G \to \mathscr{X}$ be bounded. Then X is a generalized Fourier transform of a vector measure ν on $\hat{G}$ into $\mathscr{X}$ iff for each $p \in \hat{L}^1(\hat{G})$ the mapping Y_p*

$= (Xp): G \to \mathscr{X}$ *is in* $\hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$, *i.e., iff* Y_p *is the Fourier transform of a vector measure on* $\hat{G}$ *into* $\mathscr{X}$.

Proof. Suppose X is a generalized Fourier transform of ν on $\hat{G}$ to $\mathscr{X}$, so that

$$X(g) = \int_{\hat{G}} \langle g, s\rangle \, \nu(ds), \qquad g \in G. \tag{77}$$

By hypothesis $p \in \hat{L}^1(\hat{G})$ so that $p = \hat{f}$ for a unique $f \in L^1(\hat{G})$. Hence $X(g)p(g)$ is well defined, and if $l \in \mathscr{X}^*$, then by the scalar theory one has

$$l(X(g) \cdot p(g)) = p(g)l(X(g)) = \int_{\hat{G}} \langle g, s\rangle f(s)ds \int_{\hat{G}} \langle g, t\rangle \, l \circ \nu(dt)$$

$$= \int_{\hat{G}} \langle g, s\rangle (l \circ \nu * f)ds, \text{ since } (l \circ \nu * f)\hat{} = (l \circ \nu)\hat{} \cdot \hat{f} \text{ the "}*\text{" denoting}$$

convolution,

$$= \int_{\hat{G}} \langle g, s\rangle \, k_l(s)ds, \tag{78}$$

where $k_l = l \circ \nu * f \in L^1(G)$ by the classical theory (cf. [24], p. 122 and p. 142). Also $k_{(\cdot)}(s): \mathscr{X}^* \to \mathbf{C}$ is additive, and

$$\| k_l(\cdot) \|_1 \leqslant \| f \|_1 \cdot \| l \| \cdot \| \nu \| (\hat{G}) \to 0$$

as $l \to 0$ in $\mathscr{X}^*$. Hence $k_l(s) \to 0$ as $l \to 0$ for $a \cdot a \cdot (s)$, so that $k_l(s) = \tilde{k}(s)(l)$ for a $\tilde{k}(s) \in \mathscr{X}^{**} = \mathscr{X}$ by reflexivity, and for $a \cdot a \cdot (s)$. Thus $\tilde{k}(\cdot)$ is Pettis integrable on $\hat{G}$, and the mapping $Z_p(\cdot): A \mapsto \int_A \tilde{k}(s)ds$, defines a σ-additive bounded set function into $\mathscr{X}$, a vector measure, by known results in Abstract Analysis. Consequently,

$$l(X(g)) \cdot p(g) = \int_{\hat{G}} \langle g, s\rangle \, l \circ Z_p(ds)$$

$$= l\left(\int_{\hat{G}} \langle g, s\rangle \, Z_p(ds)\right), \qquad l \in \mathscr{X}^*. \tag{79}$$

Since Z_p is a vector measure, $\| Z_p \| (\hat{G}) < \infty$, and $l \in \mathscr{X}^*$ is arbitrary, one has

$$Y_p(g) = (X \cdot p)(g) = \int_{\hat{G}} \langle g, s\rangle \, Z_p(ds) \in \mathscr{X}, \qquad g \in G, \tag{80}$$

to be well-defined. Also

$$| Y_p(g) |_{\mathscr{X}} = | p(g) | \, | X(g) |_{\mathscr{X}} \leqslant \| f \|_1 \cdot | X(g) |_{\mathscr{X}}$$

so that $\| Y_p \|_\infty \leqslant \| f \|_1 \| X \|_\infty < \infty$ and by (80) Y_p is the Fourier transform of the vector measure Z_p on $\hat{G}$ into $\mathscr{X}$. Hence $Y_p \in \hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$. This proves the direct part. The converse implication is more involved.

Thus, for the converse, let $Xp = Y_p \in \hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$ for each $p \in \hat{L}^1(\hat{G})$. Since $\mathscr{X}$ is reflexive, by Proposition 7.1, it is enough to establish that the (weakly continuous) X is V-bounded (cf. Definition 4.1). This is accomplished in two stages.

Let us first define an operator $\tau : L^1(\hat{G}) \to L^1_{\mathscr{X}}(\hat{G})$ by the equation:

$$(\tau f)\hat{} = p \cdot X = Y_p, \quad p = \hat{f}, \quad f \in L^1(\hat{G}). \tag{81}$$

Then $(\tau f)\hat{} \in \hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$ by hypothesis for each $f \in L^1(\hat{G})$. Clearly τ is linear. It is also bounded. To see this, let us show that it is closed so that the desired assertion follows by the closed graph theorem. So let $f_n, f \in L^1(\hat{G})$, $f_n \to f$ in norm, and $h_n = \tau f_n \to h$ in $\hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$. Then (cf. [21], p. 268)

$$\| \hat{f}_n - \hat{f} \|_u \leqslant \| f_n - f \|_1 \to 0 \text{ and } \| \hat{h}_n - \hat{h} \|_u \leqslant \| h_n - h \|_1 \to 0,$$

as $n \to \infty$. But then

$$\hat{h}_n = (\tau f_n)\hat{} = X \cdot \hat{f}_n \to \hat{h} \text{ and } \hat{f}_n \to \hat{f} \text{ uniformly.}$$

$$\| X\hat{f} - \hat{h} \| (s) \leqslant \| X(\hat{f}_n - \hat{f}) \| (s) + \| X\hat{f}_n - \hat{h} \| (s)$$
$$\leqslant \| X(s) \| \, | \hat{f}_n - \hat{f} | (s) + \| \hat{h}_n - \hat{h} \| (s) \to 0, \text{ as } n \to \infty, s \in \hat{G}.$$

Hence $X\hat{f} = \hat{h} = (\tau f)\hat{}$, and $\tau f = h$ (by uniqueness). So τ is closed.

Next let us verify the key property of V-boundedness for X. Since Y_p is continuous for each $p \in \hat{L}^1(\hat{G})$, it follows that X is weakly continuous. Let $h \in L^1(G)$. Consider the operator $T : L^1(G) \to \mathscr{X}$ defined by

$$T(h) = \tilde{T}(\hat{h}) = \int_G X(g)h(g)dg, \qquad \| \hat{h} \|_u \leqslant 1. \tag{82}$$

Since the correspondence $h \leftrightarrow \hat{h}$ is $1 - 1$, $\tilde{T}$ is well defined on $\hat{L}^1(G)$, and it is to be shown that $\tilde{T} : \hat{L}^1(G) \to \mathscr{X}$ is bounded when the former is endowed with the uniform norm. [Note: h below is different from h above!]

Let $h \in L^1(G)$ be arbitrarily fixed and $\{e_\alpha, \alpha \in I\} \subset L^1(\hat{G})$ be an approximate unit (cf. [24], p. 124) so that $\| e_\alpha \|_1 = 1$, $e_\alpha \geqslant 0$ and $\| (e_\alpha - e_\beta) * h \|_1 \to 0$ as $\alpha, \beta \nearrow$ "∞". Now $(\tau e_\alpha)\hat{} = X \cdot \hat{e}_\alpha$ $(= X_\alpha$, say). The hypothesis implies $X_\alpha \in \hat{\mathscr{M}}_{\mathscr{X}}(G)$, $\alpha \in I$, and

$$\| (X_\alpha - X_\beta)\hat{h} \| (t) = \| \tau(e_\alpha - e_\beta)\hat{}\, \hat{h} \| (t) \leqslant \| \tau((e_\alpha - e_\beta) * h) \|_1$$
$$\leqslant \| \tau \| \, \| (e_\alpha - e_\beta) * h \|_1 \to 0, \qquad t \in G, \tag{83}$$

since τ was shown to be bounded. Thus $X_\alpha \to X$ uniformly. Since $\hat{h} \in \hat{L}^1(G) \subset C_0(\hat{G})$ and is uniformly dense in the latter, it follows that $\| X_\alpha \|_u \leqslant C < \infty$, and the operator T_α defined below is bounded uniformly in α:

$$T_\alpha(h) = \int_G X_\alpha(t)h(t)dt, \qquad h \in L^1(G). \tag{84}$$

But X is the uniform limit of X_α's so it is also bounded, and hence T of (82) is bounded. Moreover, for $f \in C_{00}(G)$ $(\subset C_0(G))$ of compact supports,

$$\| T(hf) - T_\alpha(hf) \|_{\mathcal{X}} = \| \int_G (X - X_\alpha)(t)h(t)f(t)dt \|_{\mathcal{X}}$$
$$\leqslant \| (X - X_\alpha)f \|_u \cdot \int_G | h(t) | \, dt \to 0,$$

by (83), as $\alpha \nearrow$ "∞". Hence $\| T_\alpha(hf) \|_{\mathcal{X}} \to \| T(hf) \|_{\mathcal{X}}$, and

$$T(hf) = \lim_\alpha \int_G X_\alpha(t)h(t)f(t)dt \qquad \left(= \int_G X(t)h(t)f(t)dt\right). \tag{85}$$

If $l \in \mathcal{X}^*$, (85) implies, with $h \in L^1(G) \cap C_{00}(G) = C_{00}(G)$,

$$(l \circ T)(h) = \lim_\alpha \int_G l(X_\alpha(t))h(t)dt \qquad (= \lim_\alpha (l \circ T_\alpha)(h)).$$

On the other hand,

$$\begin{aligned}(l \circ T_\alpha) &= \int_G l(X_\alpha(t))h(t)dt = \int_G \left(l((\tau e_\alpha)\hat{})h\right)(t)dt \\ &= \int_G h(t) \cdot l\left(\int_{\hat{G}} \langle g, t\rangle (\tau e_\alpha)(g)dg\right)dt \\ &= \int_{\hat{G}} \int_G h(t) \langle g, t\rangle \, l(\tau e_\alpha)(g)dtdg, \text{ by Fubini's theorem,} \\ &= \int_{\hat{G}} l(\tau e_\alpha)(g)\hat{h}(g)dg, \text{ by Fubini again.}\end{aligned}$$

Thus for all $h \in C_{00}(g) \subset L^1(G)$,

$$| (l \circ T_\alpha)(h) | \leqslant \| \hat{h} \|_u \| l(\tau e_\alpha) \|_1 \leqslant \| \hat{h} \|_u \cdot \| l \| \| \tau \| \cdot \| e_\alpha \|_1 . \tag{86}$$

Taking suprema on $\| l \| \leqslant 1$, and noting that $\| e_\alpha \|_1 = 1$, (86) implies

$$\| T_\alpha(h) \| \leqslant \| \hat{h} \|_u \| \tau \| . \tag{87}$$

Thus (85) and (87) yield that $\| T(h) \| \leqslant c \| \hat{h} \|_u$ with $c = \| \tau \| < \infty$. Since $C_{00}(G)$ is dense in $L^1(G)$, the same holds for all $h \in L^1(G)$. So X is V-bounded. Since $\mathcal{X}$ is reflexive, Proposition 7.1 now applies and yields (77) for a unique vector measure ν on $\hat{G}$ into $\mathcal{X}$. This completes the proof.

Remark. The necessity proof also holds (and thus the theorem) if $\hat{L}^1(\hat{G})$ is replaced by

$$\hat{\mathcal{M}}(\hat{G}) = \{\hat{\mu} : \hat{\mu}(t) = \int_{\hat{G}} \langle g, t\rangle \, \mu(dg), \mu \in \mathcal{M}(\hat{G}), t \in G\},$$

where $\mathcal{M}(\hat{G})$ is the space of regular signed Borel measures on $\hat{G}$. In fact let $Y_p = \hat{\mu}X$, where $p = \hat{\mu}$ (is a function), so that for $l \in \mathcal{X}^*$,

$$\begin{aligned} l(Y_p(t)) &= \int_{\hat{G}} \langle g, t\rangle \, \mu(dg) \cdot \int_{\hat{G}} \langle s, t\rangle \, l \circ Z(ds) = (\hat{\mu} \cdot \widehat{l \circ Z})(t) \\ &= (\mu * l \circ Z)\hat{}\,(t) = l\left(\int_{\hat{G}} \langle g, t\rangle (\mu * Z)(dg)\right),\end{aligned}$$

using the convolution products appropriately (cf., e.g. [21]). Hence $\mu * Z$ is a vector measure on $\hat{G}$ and

$$\| \mu * Z \| (\hat{G}) \leqslant |\mu| (\hat{G}) \| Z \| (\hat{G}) < \infty .$$

Thus Y_p is a Fourier transform of $\mu * Z$. Identifying $L^1(\hat{G}) \hookrightarrow \mathcal{M}(\hat{G})$ as $\tilde{\mu} : A \mapsto \int_A f(t)dt$, the sufficiency proof of theorem and the above lines show that $\hat{L}^1(\hat{G})$ can be replaced by $\hat{\mathcal{M}}(\hat{G})$ every where in that result.

Taking $\mathcal{X} = L_0^2(P)$ so that V-boundedness is the same as weak harmonizability, the above theorem together with Theorems 3.3, 6.3, yield the following two summary statements on characterizations of weakly harmonizable random fields.

THEOREM 7.3. *Let G be an LCA group, $\mathcal{X} = L_0^2(P)$ be separable and $X : G \to \mathcal{X}$ be a weakly continuous mapping. Then the following statements are equivalent:*

(i) *X is weakly harmonizable.*

(ii) *X is V-bounded.*

(iii) *X is the Fourier transform of a regular vector measure on $\hat{G}$ into $\mathcal{X}$.*

(iv) *for each $p \in \hat{L}^1(\hat{G})$, the process $Y_p = Xp : G \to L_0^2(P)$ is weakly harmonizable and bounded.*

Furthermore, the following implies (i)-(iv):

(v) *if $\mathcal{H} = \overline{sp}\{X(g), g \in G\} \subset \mathcal{X}$, then there exists a weakly continuous contractive positive type family of operators $\{T(g), g \in G\} \subset B(\mathcal{H})$ such that $T(0) =$ identity, and $X(g) = T(g)X(0), g \in G$.*

In order to present a similar description of the dilation results, these individual statements should be couched in terms of classes. Let us therefore define various classes with $\mathcal{X} = L_0^2(P)$, separable.

$\mathcal{V}$ = the set of weakly continuous V-bounded random fields on G.

$\mathcal{W}$ = the set of weakly harmonizable random fields on G.

$\mathcal{F}$ = the class of all random fields which are Fourier transforms of regular vector measures on $\hat{G} \to \mathcal{X}$.

$\mathcal{M}$ = the module over $\hat{L}^1(\hat{G})$ of all functions on $G \to \mathcal{X}$ which belong to $\hat{\mathcal{M}}_{\mathcal{X}}(\hat{G})$, i.e., $\mathcal{M} = \{X : G \to \mathcal{X} \mid X \cdot \hat{L}^1(\hat{G}) \subset \hat{\mathcal{M}}_{\mathcal{X}}(\hat{G})\}$.

$\mathcal{P}$ = the class of all random fields on $G \to \mathcal{X}$ which are projections of stationary fields on $G \to \mathcal{K}$, where $\mathcal{K} \supset \mathcal{X}$ is some extension (or super) Hilbert space of $\mathcal{X}$.

Then the following result obtains:

THEOREM 7.4. *With the above notation, one has* $\mathscr{F} = \mathscr{M} = \mathscr{P} = \mathscr{V} = \mathscr{W}$

These two theorems embody essentially all the known as well as new results on the structure of weakly harmonizable processes or fields. Some applications and extensions will be indicated in the rest of the paper.

8. ASSOCIATED SPECTRA AND CONSEQUENCES

For a large class of nonstationary processes, including the (strongly) harmonizable ones, it is possible to associate a (nonnegative) spectral measure and study some of the key properties of the process through it. One such reasonably large class, isolated by Kampé de Fériet and Frankiel ([15]-[17]), called *class* (KF) in [35], is the desired family. This was also considered under the name "asymptotic stationarity" by E. Parzen [32] (cf. also [14] with the same name for a subclass), and by Rozanov ([40], p. 283) without a name. All these authors, motivated by applications, arrived at the concept independently. But it is Kampé de Fériet and Frankiel who emphasized the importance of this class and made a deep study. This was further analyzed in [35].

If $X : \mathbf{R} \to L_0^2(P)$ is a process with covariance $k(s, t) = E(X(s)\overline{X}(t))$, then it is said to be of *class* (KF), after its authors [15]-[17], provided the following limit exists for all $h \in \mathbf{R}$:

$$r(h) = \lim_{T \to \infty} \frac{1}{T} \int_0^{T-|h|} k(s, s+|h|)ds = \lim_{T \to \infty} r_T(h). \tag{88}$$

It is easy to see that $r_T(\cdot)$, hence $r(\cdot)$, is a positive definite function when $X(\cdot)$ is a measurable process. If $X(\cdot)$ is continuous in mean square, the latter is implied. It is clear that stationary processes are in class (KF). By the classical theorem of Bochner (or its modified form by F. Riesz) there is a unique bounded increasing function $F : \mathbf{R} \to \mathbf{R}^+$ such that

$$r(h) = \int_{\mathbf{R}} e^{ith} F(dt), \qquad a \cdot a \cdot (h) \cdot (\text{Leb}). \tag{89}$$

This F is termed the *associated spectral function* of the process X. Every strongly harmonizable process is of class (KF). This is not obvious, but was shown in ([40], p. 283), and in [35] as a consequence of the membership of a more general class called *almost (strongly) harmonizable*. The latter is not necessarily V-bounded and so the weakly harmonizable class is not included. (Almost