8. Associated spectra and consequences

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.07.2024**

Then the following result obtains:

THEOREM 7.4. *With the above notation, one has* $\mathcal{F} = \mathcal{M} = \mathcal{P} = \mathcal{V} = \mathcal{W}$

These two theorems embody essentially all the known as well as new results on the structure of weakly harmonizable processes or fields. Some applications and extensions will be indicated in the rest of the paper.

8. ASSOCIATED SPECTRA AND CONSEQUENCES

For a large class of nonstationary processes, including the (strongly) harmonizable ones, it is possible to associate a (nonnegative) spectral measure and study some of the key properties of the process through it. One such reasonably large class, isolated by Kampé de Fériet and Frankiel ([15]-[17]), called *class* (KF) in [35], is the desired family. This was also considered under the name "asymptotic stationarity" by E. Parzen [32] (cf. also [14] with the same name for a subclass), and by Rozanov ([40], p. 283) without a name. All these authors, motivated by applications, arrived at the concept independently. But it is Kampé de Fériet and Frankiel who emphasized the importance of this class and made a deep study. This was further analyzed in [35].

If $X : \mathbf{R} \to L_0^2(P)$ is a process with covariance $k(s, t) = E(X(s)\overline{X}(t))$, then it is said to be of *class* (KF), after its authors [15]-[17], provided the following limit exists for all $h \in \mathbf{R}$:

$$r(h) = \lim_{T\to\infty} \frac{1}{T}\int_0^{T-|h|} k(s, s+|h|)ds = \lim_{T\to\infty} r_T(h). \tag{88}$$

It is easy to see that $r_T(\cdot)$, hence $r(\cdot)$, is a positive definite function when $X(\cdot)$ is a measurable process. If $X(\cdot)$ is continuous in mean square, the latter is implied. It is clear that stationary processes are in class (KF). By the classical theorem of Bochner (or its modified form by F. Riesz) there is a unique bounded increasing function $F : \mathbf{R} \to \mathbf{R}^+$ such that

$$r(h) = \int_{\mathbf{R}} e^{ith}\, F(dt), \qquad a \cdot a \cdot (h) \cdot (\text{Leb}). \tag{89}$$

This F is termed the *associated spectral function* of the process X. Every strongly harmonizable process is of class (KF). This is not obvious, but was shown in ([40], p. 283), and in [35] as a consequence of the membership of a more general class called *almost (strongly) harmonizable*. The latter is not necessarily V-bounded and so the weakly harmonizable class is not included. (Almost harmonizable need not imply weakly harmonizable.) Since the bimeasure of (30) is not necessarily of bounded variation, the elementary proof of [40] given for the strongly harmonizable process does not extend. Perhaps for this reason, Rozanov (cf. [40], footnote on p. 283) felt that the weakly harmonizable processes may not be in class (KF). However, a positive solution can be obtained as follows:

THEOREM 8.1. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be weakly harmonizable. Then* $X \in$ class (KF), *so that it has a well defined associated spectral function.*

Proof: Since X is weakly harmonizable,

$$X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \qquad t \in \mathbf{R},$$

for a stochastic measure Z on $\mathbf{R}$ into $L_0^2(P)$, and if

$$F(A, B) = (Z(A), Z(B)),$$

then $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ is a bounded bimeasure. Considering (88) for $h \geqslant 0$ (the case $h < 0$ being similar), one has with $k(s, t) = E(X(s)\bar{X}(t))$

$$r_T(h) = \frac{T-h}{T} \cdot \frac{1}{T-h} \int_0^{T-h} k(s, s+h)ds.$$

To show that $\lim_{T\to\infty} r_T(h)$ exists it suffices to consider

$$\frac{1}{T}\int_0^T k(s, s+h)ds = \frac{1}{T}\int_0^T E(X(s) \cdot \bar{X}(s+h))ds$$

$$= E\left(\frac{1}{T}\int_0^T ds \int_{\mathbf{R}} e^{is\lambda}\, Z(d\lambda) \int_{\mathbf{R}} e^{-i(s+h)\lambda'}\, Z(d\lambda')\right) \tag{90}$$

and show that the right side has a limit as $T \to \infty$. Let $\mathscr{X} = \mathscr{Y} = L_0^2(P)$, and $\mathscr{Z} = L^1(P)$. Since $Z : \mathscr{B} \to \mathscr{X}$, $\tilde{Z} = Z : \mathscr{B} \to \mathscr{Y}$ are stochastic measures, one can define a product measure on $\mathbf{R} \times \mathbf{R}$ into $\mathscr{Z}$, using the bilinear mapping $(x, y) \to xy$, of $\mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$, as the pointwise product which is continuous in their respective norm topologies. Under these conditions and identifications, the product measure $Z \otimes \tilde{Z} : \mathscr{B} \times \mathscr{B} \to \mathscr{Z}$ is defined and satisfies (D-S integrals):

$$\int_{\mathbf{R}\times\mathbf{R}} f(s, t)\,(Z\otimes\tilde{Z})\,(ds, dt) = \int_{\mathbf{R}} Z(ds) \int_{\mathbf{R}} f(s, t)\tilde{Z}(dt)$$

$$= \int_{\mathbf{R}} \tilde{Z}(dt) \int_{\mathbf{R}} f(s, t)Z(ds), \tag{91}$$

for all $f \in C_b(\mathbf{R} \times \mathbf{R})$, by ([5], p. 388). In most of the work on product vector measures, Dinculeanu assumes that they are "dominated". However, as shown in a separate Remark (cf. [5], p. 388; cf. also [7], Cor. 3), such a product measure as in (91) is well defined even though it need not be "dominated". It has finite semivariation: indeed,

$$\| Z \otimes \tilde{Z} \| (\mathbf{R} \times \mathbf{R}) \leqslant \| Z \| (\mathbf{R}) \cdot \| \tilde{Z} \| (\mathbf{R}) = (\| Z \| (\mathbf{R}))^2 < \infty ,$$

so that $Z \otimes Z$ is again a stochastic measure. Letting

$$f_{s,h}(\lambda, \lambda') = e^{is\lambda} \cdot e^{-i(s+h)\lambda'} ,$$

so $f_{s,h} \in C_b(\mathbf{R} \times \mathbf{R})$, (91) becomes:

$$\int_{\mathbf{R}} e^{is\lambda} Z(d\lambda) \int_{\mathbf{R}} e^{-i(s+h)\lambda'} Z(d\lambda') = \int_{\mathbf{R} \times \mathbf{R}} e^{is(\lambda - \lambda') - ih\lambda'} Z \otimes Z(d\lambda, d\lambda') , \tag{92}$$

the right side being an element of $L^1(P)$. Applying the same calculation to the measures $Z \otimes Z : \mathscr{B}(\mathbf{R} \times \mathbf{R}) \to \mathscr{X}$ and $\mu : \mathscr{B}([0, T]) \to \mathbf{R}^+$ (μ is Lebesgue measure), with $(x, a) \to ax$ being the mapping of $\mathscr{X} \times \mathbf{R} \to \mathscr{X}$, one can define

$$\mu \otimes (Z \otimes Z) : \mathscr{B}(0, T) \times \mathscr{B}(\mathbf{R} \times \mathbf{R}) \to \mathscr{X}$$

and, with $\underline{\lambda}$ for the pair (λ, λ'),

$$\int_0^T \mu(dt) \int_{\mathbf{R} \times \mathbf{R}} f(t, \underline{\lambda}) Z \otimes Z(d\underline{\lambda}) = \int_{\mathbf{R} \times \mathbf{R}} Z \otimes Z(d\underline{\lambda}) \int_0^T f(t, \underline{\lambda}) \mu(dt) . \tag{93}$$

Writing $\mu(dt)$ as dt, (90)-(93) yield:

$$\begin{aligned} E\left(\frac{1}{T}\int_0^T ds \int_{\mathbf{R} \times \mathbf{R}} e^{is(\lambda - \lambda') - ih\lambda'} Z \otimes Z(d\lambda, d\lambda')\right) &= E\left(\int_{\mathbf{R} \times \mathbf{R}} e^{-ih\lambda'} Z \otimes Z(d\lambda, d\lambda') \cdot \frac{1}{T}\int_0^T e^{is(\lambda - \lambda')} ds\right) \\ &= E\left(\int_{\mathbf{R} \times \mathbf{R}} e^{-ih\lambda'} \left[\frac{e^{iT(\lambda - \lambda')} - 1}{iT(\lambda - \lambda')} \chi_{[\lambda \neq \lambda']} + \delta_{\lambda\lambda'}\right] Z \otimes Z(d\lambda, d\lambda')\right) \end{aligned} \tag{94}$$

But the quantity inside the expectation symbol E is bounded for all T, and since the dominated convergence is valid for the D-S integral ([8], IV.10.10), constants being $Z \otimes Z$-integrable, one can pass the limit as $T \to \infty$ under the expectation as well as the D-S integral in (94). Hence

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T k(s, s+h)ds = E\left(\int_{\mathbf{R}\times\mathbf{R}} e^{-ih\lambda'} \delta_{\lambda\lambda'}\, Z \otimes Z(d\lambda, d\lambda')\right)$$
$$= \int_{\mathbf{R}\times\mathbf{R}} e^{-ih\lambda'} \delta_{\lambda\lambda'}\, E(Z \otimes Z(d\lambda, d\lambda'))$$
$$= \int_{[\lambda=\lambda']} e^{-ih\lambda} F(d\lambda, d\lambda'),$$

where F is the bimeasure of Z. Hence $\lim_{T\to\infty} r_T(h) = r(h)$ exists and $r(h) = \int_{\mathbf{R}} e^{-ih\lambda}\, G(d\lambda)$, where $G : A \mapsto \int_{\pi^{-1}(A)} \delta_{\lambda\lambda'}\, F(d\lambda, d\lambda')$, $A \in \mathscr{B}$, is a positive finite measure which therefore is the associated spectral measure of $X \in$ class (KF). (Here $\pi : \mathbf{R}^2 \to \mathbf{R}$ is the coordinate projection.) This completes the proof.

The above result implies that several other considerations of [40] hold for weakly harmonizable processes.

As another application of the present work, especially as a consequence of Theorem 6.1, the following precise version of a result stated in ([40], Thm. 3.2) will be deduced from the corresponding classical stationary case.

THEOREM 8.2. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a weakly harmonizable process with* $Z : \mathscr{B} \to L_0^2(P)$ *as its representing stochastic measure. Then for any* $-\infty < \lambda_1 < \lambda_2 < \infty$, *writing* $Z(\lambda)$ *for* $Z((-\infty, \lambda))$, *one has*

$$\underset{T\to\infty}{\mathrm{l\cdot i\cdot m}} \int_{-T}^{T} \frac{e^{-it\lambda_2} - e^{-it\lambda_1}}{-it} X(t)dt$$
$$= \frac{Z(\lambda_2+) + Z(\lambda_2-)}{2} - \frac{Z(\lambda_1+) + Z(\lambda_1-)}{2} \tag{95}$$

where $\mathrm{l\cdot i\cdot m}$ *is the* $L^2(P)$*-limit. Further the covariance bimeasure* F *of* Z *can be obtained for intervals* $A = (\lambda_1, \lambda_2), B = (\lambda_1', \lambda_2')$ *as:*

$$\lim_{0\leq T_1, T_2\to\infty} \int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \frac{e^{-i\lambda_2 s} - e^{-i\lambda_1 s}}{-is}$$
$$\cdot \frac{e^{i\lambda_2' t} - e^{-i\lambda_1' t}}{it} r(s, t)dsdt = F(A, B), \tag{96}$$

provided A, B *are continuity intervals of* F *in the sense that*

$$F((-\infty, \lambda_j\pm), (-\infty, \lambda_j'\pm)) = F((-\infty, \lambda_j), (-\infty, \lambda_j')), j = 1, 2,$$

and where $r(\cdot,\cdot)$ *is the covariance function of the* X*-process. In particular, if the mapping* $S:\mathbf{R}\to\mathbf{C}$ *is continuous,* $\frac{1}{T}\int_0^T S(t)dt \to a_0$ *exists as* $T\to\infty$, *and* $\lim_{|s|+|t|\to\infty} r(s,t) = 0$, *then for the observed process* $\tilde{Y}(t) = S(t) + X(t)$, *so that* $S(\cdot)$ *is the nonstochastic "signal" and* $X(\cdot)$ *is the weakly harmonizable "noise", the estimator*

$$\hat{S}_T = \frac{1}{T}\int_0^T \tilde{Y}(t)dt \to a_0$$

in $L_0^2(P)$ *(i.e.,* $E(|\hat{S}_T - a_0|^2)\to 0$*) as* $T\to\infty$. *Thus* $\hat{S}_T$ *is a consistent estimator of* a_0, *and in other terms, both* X*- and* $\tilde{Y}$*-processes obey the law of large numbers.*

Proof: The key idea of the proof is to reduce the result to the classical stationary case through an application of the dilation theorem. Thus by Theorem 6.1, there exists a probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$, with $L_0^2(\tilde{P}) \supset L_0^2(P)$, and a stationary process $Y:\mathbf{R}\to L_0^2(\tilde{P})$ such that $X(t) = QY(t)$, $t\in\mathbf{R}$ where Q is the orthogonal projection on $L_0^2(\tilde{P})$ with range $L_0^2(P)$. There is an orthogonally scattered stochastic measure $\tilde{Z}:\mathscr{B}\to L_0^2(\tilde{P})$ such that

$$Y(t) = \int_{\mathbf{R}} e^{it\lambda}\,\tilde{Z}(d\lambda), \qquad t\in\mathbf{R}, \tag{97}$$

and $Z(A) = Q\tilde{Z}(A)$, $A\in\mathscr{B}$, with $Z:\mathscr{B}\to L_0^2(P)$ representing the given X-process. Since Q is bounded, as is well-known, it commutes with the integral as well as the l·i·m. Thus (95) is true for the Y-process with $\tilde{Z}$ in place of Z there (cf., e.g., [6], p. 527). Then the result follows on applying Q to both sides and interchanging the l·i·m as well as the integral with Q, which is legitimate. Hence (95) is true as stated.

Next consider the left hand side (LHS) of (96). With (95) it can be expressed as:

$$\text{LHS} = \lim_{T_1, T_2\to\infty} E\left(\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\left[\frac{e^{-is\lambda_2} - e^{-is\lambda_1}}{-is}X(s)\right]\cdot\left[\frac{e^{-it\lambda_2'} - e^{-it\lambda_1'}}{-it}X(t)\right]^{-} dsdt\right)$$

$$= \lim_{T_1, T_2\to\infty} E\left[\left(\int_{-T_1}^{T_1}\frac{e^{-is\lambda_2} - e^{-is\lambda_1}}{-is}X(s)ds\right)\left(\int_{-T_2}^{T_2}\frac{e^{-it\lambda_2'} - e^{-it\lambda_1'}}{-it}X(t)\right)^{-} dt\right]$$

$$= E\left[\left(\frac{Z(\lambda_2+)+Z(\lambda_2-)}{2}-\frac{Z(\lambda_1+)+Z(\lambda_1-)}{2}\right)\cdot\overline{\left(\frac{Z(\lambda_2'+)+Z(\lambda_2'-)}{2}-\frac{Z(\lambda_1'+)+Z(\lambda_1'-)}{2}\right)}\right]$$

$$= F(A, B),$$

by the continuity hypothesis on F, after expanding and taking expectations. This proves (96).

Finally, if $\tilde{Y}(t) = S(t) + X(t)$, $t \in \mathbf{R}$, let

$$a_T = E(\hat{S}_T) = \frac{1}{T}\int_0^T S(t)dt .$$

Noting that $\tilde{Y} \in$ class (KF) since X does (cf. Thm. 8.1), and $a_T \to a_0$, by hypothesis, as $T \to \infty$,

$$E(|\hat{S}_T - a_0|^2) = \frac{2}{T^2}\int_0^T\int_0^T r(s, t)dsdt + 2\,|\,a_T - a_0\,|^2$$

$$= \frac{1}{2T}\int_{-T}^T r_T(h)dh + 2\,|\,a_T - a_0\,|^2, \tag{98}$$

where, as usual, $r_T(\cdot)$ is given by (88). Since $r_T(h) \to r(h)$ due to the fact that $\tilde{Y} \in$ class (KF), and since $r(s, s+h) \to 0$ as $|\,s\,| \to \infty$ by hypothesis together with the fact that

$$|\,r(s, t\,| \leqslant \big(r(s, s)r(t, t)\big)^{1/2} \leqslant M^2 < \infty$$

where $\|\,X(t)\,\| \leqslant M < \infty$ (X being V-bounded), one can invoke a classical result on Cesarò summability (cf., [8], IV.13.83(a)). By this result $r(h) = 0$ for each $h \in \mathbf{R}$. Actually $r_T(h) \to r(h)$ $(=0)$, uniformly in h on compact sets of $\mathbf{R}$. It follows that $E(|\hat{S}_T - a_0|^2) \to 0$, and this completes the proof of the theorem.

Remark. The key reduction for (95), which is used in (96), is possible in the above proof since the linear operation of Q on the process mattered. However, for Theorem 8.1, the dilation result itself is not immediately applicable since the problem there is nonlinear, and one had to use alternate arguments as was done there. Also since Fubini's theorem is not available for the MT-integral (cf. [27], §8), a special computation has to be used for this special case. Thus the point of the general theory here is to clarify the structure of these processes, and a reduction to the stationary case is not always possible.