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Empty sums mean 0, empty products 1, and we define  $0^0 = 1$ . The notation

$$x_1 \cdots x_m / y_1 \cdots y_n$$

is sometimes used instead of

$$(x_1 \cdots x_m) (y_1 \cdots y_n)^{-1}.$$

Throughout this paper,  $E$  denotes a nonempty set of primes, to be regarded as quite arbitrary unless further assumptions are stated.  $E(x)$  is always defined by (1.2).  $p_1$  always means the smallest member of  $E$ , and if

$$E - \{p_1\} = \{p : p \in E \quad \text{and} \quad p \neq p_1\}$$

is not empty, then  $p_2$  denotes the smallest member of  $E - \{p_1\}$ . When  $x$  and  $v$  are positive, the function  $\Lambda = \Lambda(x, v; E)$  is always defined by (1.22).

### §3. PROOFS OF THEOREMS 1.6 AND 1.11, AND RELATED RESULTS

Before proving (1.8), we observe that a similar but weaker inequality has a very simple proof. For if  $y > 1$ , then

$$\log n \geq \sum_{p|n} \log p \geq \sum_{p|n, p \geq y} \log p \geq (\log y) \sum_{p|n, p \geq y} 1,$$

and hence

$$\omega(n) = \sum_{p|n, p < y} 1 + \sum_{p|n, p \geq y} 1 \leq y + (\log n) (\log y)^{-1}.$$

The right-hand side is approximately minimized by taking

$$y = (\log n) (\log_2 n)^{-2},$$

and we obtain

$$\omega(n) \leq \frac{\log n}{\log_2 n} \left\{ 1 + O\left(\frac{\log_3 n}{\log_2 n}\right) \right\} \quad \text{for} \quad n \geq 16 (> e^e). \quad (3.1)$$

Another simple proof of (3.1) can be based on Newman's observation [11, p. 652] that if  $\omega(n) = r$ , then  $n \geq r!$ .

To get the sharper inequality (1.8), it seems to be necessary to use an assumption such as (1.7) about the distribution of  $E$ . First we need a lemma relating  $\pi(x; E)$  (defined by (1.4)) and

$$\theta(x; E) = \sum_{p \leq x, p \in E} \log p. \quad (3.2)$$

LEMMA 3.3. *Suppose that there exists a real number  $\gamma(E) > 0$  such that (1.7) holds. Then for  $x > c_{11}(E)$ ,*

$$\theta(x; E) = \pi(x; E) \left\{ \log \pi(x; E) + \log_2 \pi(x; E) - 1 - \log \gamma(E) + \frac{\log_2 \pi(x; E)}{\log \pi(x; E)} + O_E \left( \frac{1}{\log \pi(x; E)} \right) \right\}. \quad (3.4)$$

*Proof:* For notational simplicity, we write  $l_r = \log_r x$ ,  $L_r = \log_r \pi(x; E)$  whenever these are defined. First note that for  $x > c_{12}(E)$ , (1.7) implies

$$L_1 = l_1 - l_2 + \log \gamma(E) + O_E(1/l_1). \quad (3.5)$$

In particular,  $L_1 \sim l_1$  and  $L_2 \sim l_2$  as  $x \rightarrow +\infty$ , so for  $x > c_{13}(E)$ , (3.5) implies

$$L_1 = l_1 \{1 + O_E(L_2/L_1)\},$$

and multiplication by  $(L_1 l_1)^{-1}$  yields

$$l_1^{-1} = L_1^{-1} \{1 + O_E(L_2/L_1)\} \quad \text{for } x > c_{13}(E). \quad (3.6)$$

Taking logarithms in (3.5), then using (3.6), we get

$$\begin{aligned} L_2 &= l_2 (1 - 1/l_1 + O_E(1/l_1 l_2)) \\ &= l_2 (1 - 1/L_1 + O_E(1/L_1 L_2)) \quad \text{for } x > c_{14}(E). \end{aligned}$$

It follows that

$$l_2 = L_2 (1 + 1/L_1 + O_E(1/L_1 L_2)) \quad \text{for } x > c_{15}(E). \quad (3.7)$$

Substituting (3.7) in (3.5), replacing  $O_E(1/l_1)$  by  $O_E(1/L_1)$ , and solving for  $l_1$ , we get

$$\begin{aligned} l_1 &= L_1 + L_2 - \log \gamma(E) + L_2/L_1 + O_E(1/L_1) \\ &\quad \text{for } x > c_{16}(E). \end{aligned} \quad (3.8)$$

We now need to estimate  $\theta(x; E)$  in terms of  $\pi(x; E)$ . We use the Stieltjes integral, then integrate by parts and combine with (1.7):

$$\begin{aligned} \theta(x; E) &= \int_1^x (\log t) d\pi(t; E) = \pi(x; E) l_1 - \int_2^x \frac{\gamma(E)}{\log t} dt \\ &\quad + O_E \left( \int_2^x \frac{dt}{(\log t)^2} \right) = \pi(x; E) l_1 - \gamma(E) (x/l_1) + O_E(x/l_1^2) \\ &= \pi(x; E) l_1 - \pi(x; E) + O_E(x/l_1^2) \end{aligned} \quad (3.9)$$

for  $x \geq 2$ . Now, (1.7) shows that

$$x/l_1^2 = O_E(\pi(x; E)/l_1) = O_E(\pi(x; E)/L_1) \quad \text{for } x > c_{17}(E).$$

Using this fact as well as (3.9) and (3.8), we get (3.4). Q.E.D.

*Proof of Theorem 1.6:* Write  $E = \{p_1, p_2, p_3, \dots\}$ , where  $p_1 < p_2 < p_3 < \dots$ . Define  $n_r = p_1 p_2 \dots p_r$  for  $r \geq 1$ . By (3.4),

$$\log n_r = \theta(p_r; E) = (r \log r) \left\{ 1 + \frac{\log_2 r}{\log r} - \frac{1 + \log \gamma(E)}{\log r} + \frac{\log_2 r}{(\log r)^2} + O_E\left(\frac{1}{(\log r)^2}\right) \right\}$$

(3.10)

for  $r > c_{11}(E)$ . Hence for  $r > c_{18}(E)$ ,

$$\log_2 n_r = \log r + \log_2 r + \frac{\log_2 r}{\log r} + O_E\left(\frac{1}{\log r}\right).$$

(3.11)

If  $r > c_{19}(E)$ , then (3.10) and (3.11) yield

$$\log n_r = r \{ \log_2 n_r - 1 - \log \gamma(E) + O_E(1/\log_2 n_r) \}.$$

(3.12)

If  $r > c_{20}(E)$ , we can solve (3.12) for  $r$  to get

$$\omega(n_r; E) = r = \frac{\log n_r}{\log_2 n_r} \left\{ 1 + \frac{1 + \log \gamma(E)}{\log_2 n_r} + O_E\left(\frac{1}{(\log_2 n_r)^2}\right) \right\}.$$

(3.13)

Now let  $n$  be any integer  $\geq 3$ , and write  $\omega(n; E) = r$ . Define

$$f(n, \alpha) = \frac{\log n}{\log_2 n} + \frac{\{1 + \log \gamma(E)\} \log n}{(\log_2 n)^2} + \alpha \frac{\log n}{(\log_2 n)^3}$$

for real  $\alpha$ . For fixed positive  $\alpha$ ,  $f(n, \alpha)$  increases with  $n$  for  $n > c_{21}(\alpha, E)$ . Thus if  $r > c_{22}(E)$ , it follows from (3.13) (since  $n \geq n_r$ ) that

$$\omega(n; E) = \omega(n_r; E) \leq f(n_r, c_{23}(E)) \leq f(n, c_{23}(E)).$$

Now suppose that  $0 \leq r = \omega(n; E) \leq c_{22}(E)$ . If  $n \geq c_{24}(E)$ , then clearly

$$f(n, c_{23}(E)) \geq c_{22}(E) \geq \omega(n; E).$$

If  $3 \leq n < c_{24}(E)$  and  $c_{25}(E)$  is sufficiently large, then (since  $\gamma(E) \leq 1$ )

$$\begin{aligned} f(n, c_{25}(E)) &\geq \frac{\log n}{(\log_2 n)^3} \left\{ c_{25}(E) + (\log \gamma(E)) \log_2 c_{24}(E) \right\} \\ &\geq c_{22}(E) \geq \omega(n; E). \end{aligned}$$

It follows that if

$$c_{26}(E) = \max \{c_{23}(E), c_{25}(E)\},$$

then

$$\omega(n; E) \leq f(n, c_{26}(E))$$

for all  $n \geq 3$ . This proves (1.8), and (3.13) shows that equality holds in (1.8) for infinitely many  $n$ . Q.E.D.

For a more precise version of (1.8) when  $E$  is the set of all primes, see [12, p. 99].

Even a much weaker hypothesis than (1.7) implies that the maximum order of  $\omega(n; E)$  is nearly  $(\log n)(\log_2 n)^{-1}$ . Specifically, suppose that there exist positive real numbers  $\delta, x_0$  such that

$$\pi(x; E) \geq x^\delta \quad \text{for all } x \geq x_0. \quad (3.14)$$

In the notation of the preceding proof, it is then clear that for  $r \geq x_0$ ,

$$\log n_r = \theta(p_r; E) \geq \pi(p_r; E) - 1 \geq p_r^\delta - 1. \quad (3.15)$$

But trivially  $\theta(p_r; E) \leq r \log p_r$ , so

$$\omega(n_r; E) = r \geq (\log n_r)(\log p_r)^{-1},$$

and hence by (3.15),

$$\limsup_{n \rightarrow +\infty} \frac{\omega(n; E) \log_2 n}{\log n} \geq \delta. \quad (3.16)$$

*Proof of Theorem 1.11:* We use the method of Erdős and Nicolas [2], which we can refine and generalize by appealing to Lemma 3.3. As before, write

$$E = \{p_1, p_2, p_3, \dots\},$$

where

$$p_1 < p_2 < p_3 < \dots$$

Assume that  $y$  satisfies (1.12) (where  $c_2(E)$  is sufficiently large), take  $r = [y] + 1$ , and let  $n_r = p_1 p_2 \dots p_r$ . There are exactly  $[x/n_r]$  multiples  $n$  of  $n_r$  such that  $n \leq x$ , and for each such  $n$  we have  $\omega(n; E) \geq r > y$ . Hence

$$S(x, y; E, \omega) \geq [x/n_r]. \quad (3.17)$$

By (3.4),

$$\log n_r = r \{ \log r + \log_2 r - 1 - \log \gamma(E) + O_E((\log_2 r)/\log r) \}$$

for  $r > c_{11}(E)$ . Define

$$g(t) = t \{ \log t + \log_2 t - 1 - \log \gamma(E) \}$$

for real  $t \geq 3$  and note that

$$0 < g'(t) = O_E(\log t) \quad \text{for } t \geq 3.$$

By the mean-value theorem for derivatives,

$$g(r) = g(y) + O_E(\log y),$$

and hence

$$\log n_r = g(y) + O_E(y(\log_2 y)/\log y) \quad \text{if } y > c_{27}(E). \quad (3.18)$$

In order to derive (1.13) from (3.17) and (3.18), we need to show that

$$[x/n_r] \gg x/n_r, \quad (3.19)$$

i.e., that  $n_r \leq x$ . For the remainder of this proof, write

$$l_k = \log_k x, \beta = 1 + \log \gamma(E) - \varepsilon,$$

and

$$z = (l_1/l_2) + \beta(l_1/l_2^2).$$

It follows from (1.12) that

$$y(\log_2 y)/\log y = O_E(l_1 l_3/l_2^2).$$

Also, if  $x > c_{28}(E, \varepsilon)$ , then

$$\log z \leq l_2 - l_3 + (\beta/l_2) \leq l_2 - l_3 + (\varepsilon/2),$$

$$\log_2 z \leq l_3.$$

It follows from these inequalities and (3.18) that if  $x > c_1(E, \varepsilon)$  (sufficiently large) and (1.12) holds, then

$$\begin{aligned} \log n_r &\leq g(z) + O_E(l_1 l_3/l_2^2) \\ &\leq (l_1/l_2)(1 + \beta/l_2) \{l_2 - l_3 + (\varepsilon/2) + l_3 - (\beta + \varepsilon)\} \\ &\quad + O_E(l_1 l_3/l_2^2) \\ &= l_1(1 - \varepsilon/2l_2) + O_{E, \varepsilon}(l_1 l_3/l_2^2) < l_1, \end{aligned}$$

so  $n_r < x$ . Thus (3.19) holds, and (1.13) follows from (3.17) and (3.18). Q.E.D.

It is interesting to observe that a result somewhat like (1.13) can be deduced from a much weaker assumption than (1.7):

THEOREM 3.20. Suppose there exist real numbers  $\delta > 0$ ,  $x_0 \geq 2$  such that (3.14) holds. If  $x \geq c_{29}(\delta)$  and  $x_0 \leq y \leq \delta (\log x) (\log_2 x)^{-1}$ , then

$$S(x, y; E, \omega) \gg x \exp \{ -\delta^{-1} (y \log y + \log y + 2) \}.$$

*Proof:* In the notation of the preceding proof, (3.17) holds, and trivially  $\log n_r \leq r \log p_r$ . If  $y \geq x_0$ , then  $p_r > r \geq x_0$  and  $r = \pi(p_r; E) \geq p_r^\delta$ , so

$$\begin{aligned} \log n_r &\leq \delta^{-1} r \log r \leq \delta^{-1} (y+1) (\log y + y^{-1}) \\ &\leq \delta^{-1} (y \log y + \log y + 2). \end{aligned} \quad (3.21)$$

But  $\log y \leq \log_2 x - \log_3 x$ , so  $\log n_r < \log x$  if  $x \geq c_{29}(\delta)$ . Hence (3.19) holds, and the result follows from (3.17) and (3.21). Q.E.D.

#### §4. PROOFS OF THEOREM 1.14 AND RELATED RESULTS

We begin by quoting the following easy result from [13, pp. 689-690]:

LEMMA 4.1. For  $x \geq 1$  and  $z \geq 1$ ,

$$\sum_{n \leq x} z^{\omega(n; E)} \leq x \prod_{p \leq x, p \in E} \{1 + (z-1) p^{-1}\}.$$

To put this in a more convenient form, we prove

LEMMA 4.2. If  $x \geq 1$  and  $w \geq -2$ , then (cf. (1.2))

$$\prod_{p \leq x, p \in E} (1 + wp^{-1}) \leq e^{wE(x)}. \quad (4.3)$$

If  $1 \leq w \leq x$ , then

$$\begin{aligned} &\prod_{p \leq x, p \in E} (1 + wp^{-1}) \\ &= \exp \{ w(E(x) - E(w)) + O(w/\log(2w)) \}. \end{aligned} \quad (4.4)$$

*Proof:* (4.3) follows immediately from the inequalities

$$0 \leq 1 + wp^{-1} \leq \exp(wp^{-1}).$$

To get (4.4), we first write

$$\begin{aligned} \prod_{p \leq x, p \in E} (1 + wp^{-1}) &\leq \prod_{p \leq w} (2wp^{-1}) \cdot \prod_{w < p \leq x, p \in E} \exp(wp^{-1}) \\ &= \exp \{ w(E(x) - E(w)) + \pi(w) \log(2w) - \theta(w) \}, \end{aligned}$$