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Artikel: THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

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Kapitel: 3.3. Notes

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Thus $c_{\xi, \lambda, m, n}$ has block matrix

$$
\begin{array}{l|ccc}
 & n\leqslant -\lambda-\frac{1}{2} & -\lambda+\frac{1}{2}\leqslant n\leqslant\lambda-\frac{1}{2} & n\geqslant\lambda+\frac{1}{2} \\
\hline
m \leqslant -\lambda - \frac{1}{2} & * & * & 0 \\
-\lambda + \frac{1}{2} \leqslant m \leqslant \lambda - \frac{1}{2} & 0 & * & 0 \\
m \geqslant \lambda + \frac{1}{2} & 0 & * & *
\end{array}
$$

where each starred block has all entries nonzero.

(d) $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leqslant m \leqslant -\lambda - \frac{1}{2}$ or $m, n \leqslant \lambda - \frac{1}{2}$ or $m, n > -\lambda + \frac{1}{2}$. □

The finite-dimensional representation occurring in the above classification are the representations $\pi^0_{\xi, \lambda}(\lambda+\xi\in\mathbf{Z}+\frac{1}{2}, \lambda\neq 0)$.

3.3. Notes

3.3.1. In the case of the unitary principal series (λ imaginary), Theorem 3.4 was first proved by Bargmann [2, sections 6 and 7]. See van Dijk [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by Barut & Phillips [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero λ is contained in a general theorem by Bruhat [5, Theorem 7; 2]: For $\xi \in \hat{M}$, $\lambda \in i\mathfrak{a}$, the principal series representation $\pi_{\xi, \lambda}$ of G (cf. (2.2)) is irreducible if $s \,.\, \lambda \neq \lambda$ for all $s \neq e$ in the Weyl group for (G, K).

3.3.3. Gelfand & Naimark [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for $SL(2, \mathbf{C})$ by a global method different from ours, working in a noncompact realization and calculating the "matrix elements" of the representation with respect to a (continuous) $\overline{N}$-basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian K, cf. [27, Theorem 3.3]. In that case the canonical matrix elements $\tau_{\gamma, \delta}$ are matrix-valued functions. By using this method, Naimark [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for $SL(2, \mathbf{C})$, see also Kosters [28].

3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of $\mathbf{R}^2$ and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K-multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K-multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G. The representation σ is called *Naimark related* to τ if there is a closed (possibly) unbounded) injective linear operator A from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ with domain $\mathscr{D}(A)$ dense in $\mathscr{H}(\sigma)$ and range $\mathscr{R}(A)$ dense in $\mathscr{H}(\tau)$ such that $\mathscr{D}(A)$ is σ-invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathscr{D}(A)$, $g \in G$. Then we use the notation $\sigma \underset{A}{\simeq} \tau$ or $\sigma \simeq \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K-multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathscr{H}(\sigma)$ onto $\mathscr{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathscr{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.