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Autor: Koornwinder, Tom H.
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Thus $c_{\xi, \lambda, m, n}$ has block matrix

$$\begin{array}{ccc}
 n \leq -\lambda - \frac{1}{2} & -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2} & n \geq \lambda + \frac{1}{2} \\
 \begin{array}{l} m \leq -\lambda - \frac{1}{2} \\ -\lambda + \frac{1}{2} \leq m \leq \lambda - \frac{1}{2} \\ m \geq \lambda + \frac{1}{2} \end{array} & \left(\begin{array}{ccc} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{array} \right) &
 \end{array}$$

where each starred block has all entries nonzero.

(d) $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leq m \leq -\lambda - \frac{1}{2}$ or $m, n \leq \lambda - \frac{1}{2}$
 or $m, n > -\lambda + \frac{1}{2}$. □

The finite-dimensional representation occurring in the above classification are the representations $\pi_{\xi, \lambda}^0(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0)$.

3.3. NOTES

3.3.1. In the case of the unitary principal series (λ imaginary), Theorem 3.4 was first proved by BARGMANN [2, sections 6 and 7]. See van DIJK [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by BARUT & PHILLIPS [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero λ is contained in a general theorem by BRUHAT [5, Theorem 7; 2]: For $\xi \in \hat{M}$, $\lambda \in ia$, the principal series representation $\pi_{\xi, \lambda}$ of G (cf. (2.2)) is irreducible if $s \cdot \lambda \neq \lambda$ for all $s \neq e$ in the Weyl group for (G, K) .

3.3.3. GELFAND & NAIMARK [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for $SL(2, \mathbf{C})$ by a global method different from ours, working in a noncompact realization and calculating the “matrix elements” of the representation with respect to a (continuous) \overline{N} -basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian K , cf. [27, Theorem 3.3]. In that case the canonical matrix elements $\tau_{\gamma, \delta}$ are matrix-valued functions. By using this method, NAIMARK [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for $SL(2, \mathbf{C})$, see also KOSTERS [28].

3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of \mathbf{R}^2 and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K -multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K -multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G . The representation σ is called *Naimark related* to τ if there is a closed (possibly unbounded) injective linear operator A from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\tau)$ with domain $\mathcal{D}(A)$ dense in $\mathcal{H}(\sigma)$ and range $\mathcal{R}(A)$ dense in $\mathcal{H}(\tau)$ such that $\mathcal{D}(A)$ is σ -invariant and $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathcal{D}(A)$, $g \in G$. Then we use the notation $\sigma \stackrel{A}{\simeq} \tau$ or $\sigma \simeq \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K -multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathcal{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.