**Zeitschrift:** L'Enseignement Mathématique

**Band:** 28 (1982)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

**Kapitel:** 5. Equivalence of irreducible representations of SU(1, 1) to subrepresentations of the principal series

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**DOI:** https://doi.org/10.5169/seals-52233

## 5. Equivalence of irreducible representations of $SU(1, 1)$ to subrepresentations of the principal series

The first two subsections review some generalities about Gelfand pairs and spherical functions. By using the concepts developed there we can next, in §5.4, translate the problem of classifying the irreducible representations of $SU(1, 1)$ in such a way that the problem can be solved by global methods. For this the generalized Abel transform (§5.3) and the Chebyshev transform pair of Deans (Theorem 5.10) are the main tools. The problem is finally reduced to finding the continuous characters on the convolution algebra $\mathscr{D}_{\text{even}}(\mathbf{R})$ (Prop. 5.7).

### 5.1. Spherical functions

We remember some of the standard facts about spherical functions (cf. for instance Godement [20], Helgason [25, Ch. X], Faraut [12, Ch. 1]). Let $G$ be a unimodular lcsc. group with compact subgroup $K$. $(G, K)$ is called a *Gelfand pair* if $C_c(K\backslash G/K)$ is a commutative algebra under convolution. If there is a continuous involutive automorphism $\alpha$ on $G$ such that $\alpha(KxK) = Kx^{-1}K (x \in G)$ then $(G, K)$ is a Gelfand pair. If $(G, K)$ is a Gelfand pair and the irreducible representation $\tau$ of $G$ is unitary or $K$-finite then the representation 1 of $K$ has multiplicity 0 or 1 in $\tau$.

Let $(G, K)$ be a Gelfand pair. A *spherical function* is a function $\phi \neq 0$ on $G$ such that

$$\phi(x)\phi(y) = \int_K \phi(xky)dk, \quad x, y \in G.$$

The nonzero continuous algebra homomorphisms from $C_c(K\backslash G/K)$ (or $C_c^\infty(K\backslash G/K)$ if $G$ is a Lie group) to $\mathbf{C}$ are precisely of the form

$$f \to \int_G f(x)\phi(x^{-1})dx, \tag{5.1}$$

where $\phi$ is a spherical function. If $\tau$ is a $K$-unitary representation of $G$ and if $\mathscr{H}(\tau)$ contains a $K$-fixed unit vector $v$, unique up to a constant factor, then $x \to (\tau(x)v, v)$ is a spherical function.

## 5.2. Spherical functions of type $\delta$

Let $G$ be a unimodular lcsc. group with compact subgroup $K$. Let

$$K^* := \{(k, k) \in G \times K \mid k \in K\}.$$

Let $\delta \in \hat{K}$ and let $\tau$ be a $K$-unitary representation of $G$. Then $\tau \otimes \check{\delta}$ ($\check{\delta}$ the contragredient representation to $\delta$) is a $K^*$-unitary representation of $G \times K$ on $\mathscr{H}(\tau) \otimes \mathscr{H}(\delta)$.

LEMMA 5.1. *The multiplicity of* $\delta$ *in* $\tau|_K$ *is equal to the multiplicity of the representation* 1 *of* $K^*$ *in* $\tau \otimes \check{\delta}|_{K^*}$. $\tau$ *is irreducible iff* $\tau \otimes \check{\delta}$ *is irreducible.* $\tau$ *is unitary iff* $\tau \otimes \check{\delta}$ *is unitary.*

This can be proved immediately. By using the results summarized in §5.1 we conclude that $(G \times K, K^*)$ is a Gelfand pair if there exists a continuous involutive homomorphism $\alpha$ on $G$ such that for each $(g, k) \in G \times K$ we have $\alpha(g) = k_1 g^{-1} k_2$, $\alpha(k) = k_1 k^{-1} k_2$ for certain $k_1, k_2 \in K$. Furthermore, if $(G \times K, K^*)$ is a Gelfand pair and if the irreducible representation $\tau$ of $G$ is unitary or $K$-finite then $\tau$ is $K$-multiplicity free. In particular, this applies to $SU(1, 1)$:

PROPOSITION 5.2. *If* $G = SU(1, 1)$ *then* $(G \times K, K^*)$ *is a Gelfand pair.*

*Proof.* For $g \in SU(1, 1)$ define $\alpha(g) := {}^t(g^{-1})$. Then $\alpha$ is a continuous involutive automorphism on $G$ and $\alpha(a_t) = a_{-t}$ on $A$, $\alpha(u_\theta) = u_{-\theta}$ on $K$. Since $G = KAK$, $\alpha$ has the required properties. □

Let $(G \times K, K^*)$ be a Gelfand pair. Identify $G \times \{e\}$ with $G$. A spherical function on $G \times K$ is completely determined by its restriction to $G$. By using the results mentioned in §5.1 we obtain the following properties. First, a continuous function $\phi$ on $G$ is the restriction to $G$ of a spherical function on $G \times K$ iff $\phi \neq 0$ and

$$\phi(x)\phi(y) = \int_K \phi(xkyk^{-1})dk, \qquad x, y \in G.$$

Next, let

$$I_c(G) \ (\text{or } I_c^\infty(G)) := \{f \in C_c(G) \ (\text{or } C_c^\infty(G)) \mid f(kgk^{-1}) = f(g), \ g \in G, k \in K\}.$$

These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where $\phi$ is a spherical function on $G \times K$. If $\phi$ is a spherical function on $G \times K$ then there is a $\delta \in \hat{K}$ such that for all $x \in G$ the function $k \to \phi(xk)$ on $K$ belongs to $\delta$. Then $\delta$ is called a *spherical function of type* $\delta$ on $G$ (with respect to $K$), cf. GODEMENT [19]. It is funny that spherical functions of type $\delta$ are on the one hand generalizations of ordinary spherical functions for $(G, K)$, on the other hand restrictions to $G$ of ordinary spherical functions for $(G \times K, K^*)$.

For convenience, we take a one-dimensional $\delta \in \hat{K}$. Then a spherical function $\phi$ on $G \times K$ is of type $\delta$ iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \qquad x \in G, k \in K .$$

Let

$$\begin{aligned} &I_{c,\delta}(G) \ \left(\text{or } I^{\infty}_{c,\delta}(G)\right) \\ &:= \left\{f \in C_c(G) \ \left(\text{or } C_c^{\infty}(G)\right) \mid f(xk) = f(kx) \right. \\ &\qquad \left. = \delta(k)f(x),\ x \in G,\ k \in K\right\} . \end{aligned}$$

These are closed subalgebras of $I_c(G)$ (or $I_c^{\infty}(G)$) and their characters are precisely of the form (5.1), where $\phi$ is a spherical function of type $\delta$. Finally, if $\tau$ is a $K$-unitary representation of $G$ and if $\mathscr{H}(\tau)$ contains a unit vector $v$ satisfying $\tau(k)v = \delta(k)v$, unique up to a constant factor, then $x \to (\tau(x)v, v)$ is a spherical function of type $\delta$.

## 5.3. THE GENERALIZED ABEL TRANSFORM

Let $G$ be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures $dk, da, dn$ on $K, A, N$, respectively, normalize the Haar measure on $G$ such that

$$(5.2) \qquad \int_G f(g)dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)}\, dk\, da\, dn,\ f \in C_c(G)$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$(5.3) \qquad \int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn,\ f \in C_c(N),\ a \in A$$

(cf. [25, Ch. X, proof of Prop. 1.11]).

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let $U^\lambda$ be the representation of $G$ induced by the one-dimensional representation $an \to e^{\lambda(\log a)}$ of the subgroup $AN$:

(5.4) $(U^\lambda(g)f)(k) := e^{-(\rho+\lambda)H(g^{-1}k)} f(u(g^{-1}k)),\ f \in L^2(K),\ g \in G,\ k \in K$.

The representation $U^\lambda$ is easily seen to split as a direct sum of principal series representations $\pi_{\xi,\lambda}$. $U^\lambda$ restricted to $K$ is the left regular representation of $K$.

Let $\delta \in \hat{K}$. For convenience, suppose that $\delta$ is one-dimensional. The *generalized Abel transform* $f \to F_f^\delta : I_{c,\delta}(G) \to C_c(A)$ is defined by

$$F_f^\delta(a) := e^{\rho(\log a)} \int_N f(an)dn,\ a \in A. \tag{5.5}$$

If $G = SU(1, 1)$ and $\delta = 1$ then this transform can be rewritten as the classical Abel transform, cf. §5.4.

PROPOSITION 5.3. *The mapping* $f \to F_f^\delta$ *is a continuous homomorphism (with respect to convolution on* $G$ *and* $A$, *respectively) from* $I_{c,\delta}^\infty(G)$ *to* $C_c^\infty(A)$. *Furthermore,*

$$\int_A F_f^\delta(a)e^{-\lambda(\log a)}\,da = \int_G f(g)(U^\lambda(g^{-1})\check{\delta}, \check{\delta})dg,\ f \in I_{c,\delta}^\infty(G),\ \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \tag{5.6}$$

*where* $(.,.)$ *denotes the inner product on* $L^2(K)$.

*Proof.* The continuity is immediate. The homomorphism property follows easily from (5.2) and (5.3) (cf. WARNER [49, pp. 34, 35]). For the proof of (5.6) substitute (5.4) into the right hand side of (5.6):

$$\begin{aligned}\int_G f(g)(U^\lambda(g^{-1})\check{\delta}, \check{\delta})dg &= \int_G \int_K f(g)e^{-(\rho+\lambda)H(gk)}\,\check{\delta}((u(gk))^{-1}k)dk\,dg \\ &= \int_G f(g)e^{-(\rho+\lambda)H(g)}\,\check{\delta}((u(g))^{-1})dg \\ &= \int_{K\times A\times N} f(kan)e^{(\rho-\lambda)\log a}\,\check{\delta}(k^{-1})dk\,da\,dn \\ &= \int_A \int_N f(an)e^{(\rho-\lambda)\log a}\,dn\,da \\ &= \int_A F_f^\delta(a)e^{-\lambda(\log a)}\,da.\end{aligned}$$

□

Now let $G = SU(1, 1)$. Write $F_f^n(t)$ and $I_{c,n}^\infty(G)$ instead of $F_f^{\delta_n}(a_t)$ and $I_{c,\delta_n}^\infty(G)$, respectively. If $n \in \mathbf{Z} + \xi$ then (5.5) and (5.6) take the form

$$F_f^n(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_z) dz \tag{5.7}$$

and

$$\int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t}\, dt = \int_G f(g) \pi_{\xi,\lambda,n,n}(g^{-1}) dg, \; f \in I_{c,n}^\infty(G), \lambda \in \mathbf{C}\,, \tag{5.8}$$

where $dg = (2\pi)^{-1} e^t d\theta dt dz$ if $g = u_\theta a_t n_z$.

## 5.4. The main theorem

It is the purpose of this section to prove:

THEOREM 5.4. *Let* $\tau$ *be an irreducible* $K$*-unitary representation of* $SU(1, 1)$ *which is* $K$*-finite or unitary. Then* $\tau$ *is Naimark equivalent to an irreducible subrepresentation of some principal series representation* $\pi_{\xi,\lambda}$.

By Proposition 5.2 $\tau$ is $K$-multiplicity free. If $\delta_n \in \mathscr{M}(\tau)$ then write $\tau_{n,n}$ instead of $\tau_{\delta_n,\delta_n}$. In view of Theorem 4.5 and Remark 4.8 it is sufficient for the proof of Theorem 5.4 to show that for some $\delta_n \in \mathscr{M}(\tau)$, for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ with $n \in \mathbf{Z} + \xi$ we have

$$\tau_{n,n} = \pi_{\xi,\lambda,n,n}\,. \tag{5.9}$$

Both sides of (5.9) are spherical functions of type $\delta_n$. Then (5.9) holds if the corresponding characters on $I_{c,n}^\infty(G)$ are equal. Hence Theorem 5.4 will follow from

PROPOSITION 5.5. *Let* $G = SU(1, 1)$, $n \in \frac{1}{2}\mathbf{Z}$. *Let* $\alpha$ *be a continuous character on* $I_{c,n}^\infty(G)$. *Then*

$$\alpha(f) = \int_G f(g) \pi_{\xi,\lambda,n,n}(g^{-1}) dg, \; f \in I_{c,n}^\infty(G)\,, \tag{5.10}$$

*for some* $\lambda \in \mathbf{C}$ *and for* $\xi \in \{0, \frac{1}{2}\}$ *such that* $n \in \mathbf{Z} + \xi$.

Now substitute (5.8) into the right hand side of (5.10). Thus, for the proof of Prop. 5.5 we have to show that each continuous character $\alpha$ on $I^{\infty}_{c,n}(G)$ takes the form

$$\alpha(f) = \int_{-\infty}^{\infty} F^n_f(t) e^{-\lambda t}\, dt, \quad f \in I^{\infty}_{c,n}(G)\,. \tag{5.11}$$

for some $\lambda \in \mathbf{C}$. In §5.5 we will prove:

THEOREM 5.6. *Let* $G = SU(1, 1)$, $n \in \frac{1}{2}\mathbf{Z}$. *The mapping* $f \to F^n_f$ *is a topological algebra isomorphism from* $I^{\infty}_{c,n}(G)$ *onto* $\mathscr{D}_{\text{even}}(\mathbf{R})$, *the algebra of even* $C^{\infty}$*-functions with compact support on* $\mathbf{R}$.

Thus, in view of (5.11) we are left to prove:

PROPOSITION 5.7. *The continuous characters on* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *have the form*

$$h \to \int_{-\infty}^{\infty} h(t) e^{-\lambda t}\, dt$$

*for some* $\lambda \in \mathbf{C}$.

## 5.5. COMPLETION OF THE PROOF OF THE MAIN THEOREM

By the discussion in §5.4 we reduced the proof of Theorem 5.4 to the task of proving Theorem 5.6 and Prop. 5.7. Theorem 5.6 was partly proved in Prop. 5.3. It is left to prove that $f \to F^n_f$ is injective on $I^{\infty}_{c,n}(G)$ with image $\mathscr{D}_{\text{even}}(\mathbf{R})$ and that the inverse mapping is continuous. In order to establish this we identify both $I^{\infty}_{c,n}(G)$ and $\mathscr{D}_{\text{even}}(\mathbf{R})$, considered as topological vector spaces, with $\mathscr{D}([1, \infty))$ and we rewrite $f \to F^n_f$ as a mapping from $\mathscr{D}([1, \infty))$ onto itself. This mapping turns out to be a known integral transformation, for which an inverse transformation can be explicitly given. First note:

LEMMA 5.8. *The formula*

$$f(x) = g(x^2) \tag{5.12}$$

*defines an isomorphism of topological vector spaces* $f \to g$ *from* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *onto* $\mathscr{D}([0, \infty))$.

*Proof.* Clearly, if $g \in \mathscr{D}([0, \infty))$ then $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$ and the mapping $g \to f$ is continuous. Conversely, let $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$ and let $g$ be defined by (5.12). By complete induction with respect to $n$ we prove: $g^{(n)}(0)$ exists and there is a function $f_n \in \mathscr{D}_{\text{even}}(\mathbf{R})$ such that

$$f_n(x) = g^{(n)}(x^2),\ x \in \mathbf{R}\,,$$

and $f \to f_n : \mathscr{D}_{\text{even}}(\mathbf{R}) \to \mathscr{D}_{\text{even}}(\mathbf{R})$ is continuous. Indeed, suppose this is proved up to $n - 1$. Then

$$2x(g^{(n-1)})'(x^2) = f'_{n-1}(x) = \int_0^x f''_{n-1}(y)dy\,,$$

so

$$g^{(n)}(x^2) = \tfrac{1}{2}\int_0^1 f''_{n-1}(tx)dt =: f_n(x)\,. \qquad \square$$

For $f \in I^{\infty}_{c,n}(G)$ define

$$\tilde{f}(x) := f\left(\begin{pmatrix} x & (x^2-1)^{\frac{1}{2}} \\ (x^2-1)^{\frac{1}{2}} & x \end{pmatrix}\right),\ x \in [1, \infty)\,. \tag{5.13}$$

For $f \in I^{\infty}_{c,n}(G)$ define

$$\tilde{h}(ch\tfrac{1}{2}t) := h(t),\ t \in \mathbf{R}\,. \tag{5.14}$$

LEMMA 5.9. *The mapping* $f \to \tilde{f}$ *defined by* (5.13) *is an isomorphism of topological vector spaces from* $I^{\infty}_{c,n}(G)$ *onto* $\mathscr{D}([1, \infty))$. *The mapping* $h \to \tilde{h}$ *defined by* (5.35) *is an isomorphism of topological vector spaces from* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *onto* $\mathscr{D}([1, \infty))$.

*Proof.* The second statement follows from Lemma 5.8. For the proof of the first statement introduce global real analytic coordinates on $G$ by the mapping

$$(z, \phi) \to \begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}$$

from $C \times (\mathbf{R}/4\pi\mathbf{Z})$ onto $G$. If $g \in \mathscr{D}([1, \infty))$ and

$$f\left(\begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}\right) := e^{in\phi}g((1+|z|^2)^{\frac{1}{2}})$$

then $f \in I^{\infty}_{c,n}(G)$, $\tilde{f} = g$ and the mapping $g \to f$ is continuous. Conversely, if $f \in I^{\infty}_{c,n}(G)$ then $f$, as a function of $z$ and $\phi$, is radial in $z$, so the function

$$z \to f\begin{pmatrix} (1+z^2)^{\frac{1}{2}} & z \\ z & (1+z^2)^{\frac{1}{2}} \end{pmatrix}, \; z \in \mathbf{R},$$

belongs to $\mathscr{D}_{\text{even}}(\mathbf{R})$. Now make the transformation $z = (x^2-1)^{\frac{1}{2}}$ and apply Lemma 5.8. It follows that $\tilde{f} \in \mathscr{D}([1, \infty))$ and that the mapping $f \to \tilde{f}$ is continuous. □

Define the Chebyshev polynomial $T_n(x)$ by

$$T_n(\cos\theta) := \cos n\theta. \tag{5.15}$$

It follows from (5.7) that, for $f \in I^{\infty}_{c,n}(G)$:

$$\begin{aligned} F^n_f(t) &= e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f\left(\begin{pmatrix} ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t} & * \\ * & * \end{pmatrix}\right) dz \\ &= e^{\frac{1}{2}t} \int_{-\infty}^{\infty} \tilde{f}(|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|) \left(\frac{ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}}{|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|}\right)^{2n} dz \\ &= e^{\frac{1}{2}t} \int_{0}^{\infty} \tilde{f}(|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|) T_{2|n|}\left(\frac{ch\tfrac{1}{2}t}{|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|}\right) dz, \end{aligned}$$

so

$$F^n_f(t) = 2 \int_{ch\frac{1}{2}t}^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1} ch\tfrac{1}{2}t)(y^2 - ch^2\tfrac{1}{2}t)^{-\frac{1}{2}} y\,dy.$$

This formula shows that $F^n_f$ is even on $\mathbf{R}$, so $F^n_f \in \mathscr{D}_{\text{even}}(\mathbf{R})$. Now, by (5.14):

$$\tilde{F}^n_f(x) = 2 \int_{x}^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1}x)(y^2 - x^2)^{-\frac{1}{2}} y\,dy, \; x \in [1, \infty]. \tag{5.16}$$

For $n = 0$, (5.16) takes the form

$$\tilde{F}^0_f(x) = 2 \int_{x}^{\infty} \tilde{f}(y)(y^2 - x^2)^{-\frac{1}{2}} y\,dy:$$

The problem of inverting this just means to solve the Abel integral equation, as was pointed out by GODEMENT [20]. Indeed, we get

$$\tilde{f}(y) = -\pi^{-1} \int_y^\infty \frac{d}{dx} \tilde{F}_f^0(x)(x^2-y^2)^{-\frac{1}{2}} dx .$$

For general $n$, we can use an inversion formula obtained by DEANS [7, (30)], see also MATSUSHITA [30, §2.3] and KOORNWINDER [27, §5.9]:

THEOREM 5.10. *For* $m = 0, 1, 2, \ldots, g \in \mathscr{D}([1, \infty)), x \in [1, \infty)$ *define*

(5.17) $$(A_m g)(x) := 2 \int_x^\infty g(y) T_m (y^2 - x^2)^{-\frac{1}{2}} y dy ,$$

(5.18) $$(B_m g)(x) := -\pi^{-1} \int_x^\infty g'(y) T_m(x^{-1}y)(y^2-x^2)^{-\frac{1}{2}} dy .$$

*Then* $A_m$ *and* $B_m$ *map* $\mathscr{D}([1, \infty))$ *into itself and* $A_m B_m = id, B_m A_m = id.$

This theorem shows that $f \to F_f^n$ is a linear bijection from $I_{c,n}^\infty(G)$ onto $\mathscr{D}_{\text{even}}(\mathbf{R})$. Finally in order to prove the continuity of the inverse mapping, we show that $B_m$ is continuous. Just expand $T_m(x^{-1}y)$ as a polynomial and use that

$$\left(x^{-1} \frac{d}{dy}\right)^p \int_x^\infty h(y)(y^2-x^2)^{-\frac{1}{2}} y dy$$

$$= \int_x^\infty \left(y^{-1} \frac{d}{dy}\right)^p h(y)(y^2-x^2)^{-\frac{1}{2}} y dy$$

by the properties of the Weyl fractional integral transform (cf. [11, Ch. 13]). This completes the proof of Theorem 5.6.

*Proof of proposition* 5.7. Extend $\alpha$ to a continuous linear functional on $\mathscr{D}(\mathbf{R})$, for instance by putting $\alpha(f) = 0$ if $f$ is odd. Choose $f_1 \in \mathscr{D}_{\text{even}}(\mathbf{R})$ such that $\alpha(f_1) \neq 0$. Let

$$(\lambda(y) f_1)(x) := f_1(x-y),\ x, y \in \mathbf{R} .$$

By the continuity and homomorphism property of $\alpha$ we have, for $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$:

$$\alpha(f_1)\alpha(f) = \alpha(f_1 * f) = \int_{-\infty}^{\infty} \alpha(\lambda(y)f_1)f(y)dy\,.$$

Hence

$$\alpha(f) = \int_{-\infty}^{\infty} f(y)\beta(y)dy,\ f \in \mathscr{D}_{\text{even}}(\mathbf{R})\,,$$

where

$$\beta(y) := \tfrac{1}{2}(\alpha(f_1))^{-1}(\alpha(\lambda(y)f_1) + \alpha(\lambda(-y)f_1))\,.$$

Then $\beta$ is even and it is a continuous function by the continuity of $\alpha$. It follows from the homomorphism property of $\alpha$ and from the fact that $\beta$ is even, that

$$\beta(x)\beta(y) = \tfrac{1}{2}(\beta(x+y) + \beta(x-y))\,,$$

so $\beta(0) = 1$. This is d'Alembert's functional equation. By continuity, $Re\ \beta(x) > 0$ if $0 \leqslant x \leqslant x_0$ for some $x_0 > 0$. Then $\beta(x_0) = \cosh c$ for some complex $c = a + ib$ with $a \geqslant 0$, $-\tfrac{1}{2}\pi < b < \tfrac{1}{2}\pi$. Now, following the proof in ACZEL [1, 2.4.1] it can be shown [1]) that for all integer $n, m \geqslant 0$

$$\beta\left(\frac{n}{2^m}x_0\right) = \cosh\left(\frac{c}{x_0}\frac{n}{2^m}x_0\right).$$

So, by continuity and evenness of $\beta$:

$$\beta(x) = \cosh\left(\frac{c}{x_0}x\right) \text{ for all } x \in \mathbf{R}\,. \qquad \square$$

## 5.6. NOTES

5.6.1. Some other examples of Gelfand pairs $(G \times K, K^*)$ are provided by $G = SO_0(n, 1)$, $K = SO(n)$ and $G = SU(n, 1)$, $K = S(U(n) \times U(1))$, cf. BOERNER [4, Ch. VII, §12; Ch. V, §6], DIXMIER [8] or KOORNWINDER [27, Theorems 5.7, 5.8].

5.6.2. The main Theorem 5.4, which was first proved in the case of unitary representations by BARGMANN [2], is a special case of the *subrepresentation*

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[1]) I thank H. van Haeringen for this reference.

*theorem* for noncompact semisimple Lie groups due to Casselman (cf. WALLACH [47, Cor. 7.5]). Casselman's theorem improves HARISH-CHANDRA'S [22, Theorem 4] *subquotient theorem.*

5.6.3. The generalized Abel transform $f \to F_f^\delta$ can be defined for general $K$-type $\delta$. It was introduced by HARISH-CHANDRA [24, p. 595] in the spherical case, TAKAHASHI [40, §2] in the case $G = SO_0(n, 1)$ and WARNER [49, 6.2.2] in the general case. The injectivity of this transform holds generally, cf. WARNER [49]. The image of $I_{c,\delta}^\infty(G)$ under this transform is known in the spherical case (cf. GANGOLLI [16]) and if $G$ has real rank 1 and $\delta$ is one-dimensional (cf. WALLACH [46]), but seems to be unknown in the general case (cf. WARNER [49, p. 36]).

5.6.4. In [39] TAKAHASHI also reduces the proof of Theorem 5.4 to Proposition 5.5. However, he proves Prop. 5.5 by considering eigenfunctions of the Casimir operator, since he did not know, then, how to invert the transform $f \to F_f^n$. In [42] he independently obtained a proof of Prop. 5.5 similar to ours. Earlier, in [40, §4.1] he used a similar method in the spherical case of $G = SO_0(n, 1)$. NAIMARK [34, Ch. 3, §9] proved the subquotient theorem for $SL(2, \mathbf{C})$ by methods somewhat related to ours.

5.6.5. Part of Lemma 5.8 is contained in WHITNEY [50]. See SCHWARZ [37] for a theorem on $C^\infty$-functions which are invariant under a more general Weyl group.

5.6.6. Theorem 5.10 more generally holds with Gegenbauer polynomials of integer of half integer order as kernels, cf. DEANS [6], [7], KOORNWINDER [27, §5.9]. Deans' proof uses the inversion formula for the Radon transform. The author's proof uses Weyl fractional integral transforms and generalized fractional integral transforms studied by SPRINKHUIZEN [38]. MATSUSHITA [30, §2.3] considers the transformation $f \to F_f^n$ for general real $n$ in the context of the universal covering group of $SL(2, \mathbf{R})$ and he derives the inversion formula with a proof due to T. Shintani, which uses Mellin transforms.

## 6. UNITARIZABILITY OF IRREDUCIBLE SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

### 6.1. A CRITERIUM FOR UNITARIZABILITY

Remember that a representation of an lcsc. group $G$ on a Hilbert space is strongly continuous if and only if it is weakly continuous (cf. WARNER [48, Prop. 4.2.2.1]). Thus, if $\tau$ is a (strongly continuous) Hilbert representation of $G$ then $\tilde{\tau}$ defined by