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Kapitel: 5.4. The main theorem

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Now let $G = SU(1, 1)$. Write $F_f^n(t)$ and $I_{c,n}^{\infty}(G)$ instead of $F_f^{\delta_n}(a_t)$ and $I_{c,\delta_n}^{\infty}(G)$, respectively. If $n \in \mathbf{Z} + \xi$ then (5.5) and (5.6) take the form

$$F_f^n(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_z) dz \tag{5.7}$$

and

$$\int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t} \, dt = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \; f \in I_{c,n}^{\infty}(G), \lambda \in \mathbf{C}, \tag{5.8}$$

where $dg = (2\pi)^{-1} e^t d\theta dt dz$ if $g = u_\theta a_t n_z$.

5.4. The main theorem

It is the purpose of this section to prove:

THEOREM 5.4. *Let τ be an irreducible K-unitary representation of $SU(1, 1)$ which is K-finite or unitary. Then τ is Naimark equivalent to an irreducible subrepresentation of some principal series representation $\pi_{\xi, \lambda}$.*

By Proposition 5.2 τ is K-multiplicity free. If $\delta_n \in \mathscr{M}(\tau)$ then write $\tau_{n,n}$ instead of $\tau_{\delta_n, \delta_n}$. In view of Theorem 4.5 and Remark 4.8 it is sufficient for the proof of Theorem 5.4 to show that for some $\delta_n \in \mathscr{M}(\tau)$, for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ with $n \in \mathbf{Z} + \xi$ we have

$$\tau_{n,n} = \pi_{\xi, \lambda, n, n} \, . \tag{5.9}$$

Both sides of (5.9) are spherical functions of type δ_n. Then (5.9) holds if the corresponding characters on $I_{c,n}^{\infty}(G)$ are equal. Hence Theorem 5.4 will follow from

PROPOSITION 5.5. *Let $G = SU(1, 1)$, $n \in \frac{1}{2}\mathbf{Z}$. Let α be a continuous character on $I_{c,n}^{\infty}(G)$. Then*

$$\alpha(f) = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \; f \in I_{c,n}^{\infty}(G), \tag{5.10}$$

for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ such that $n \in \mathbf{Z} + \xi$.

Now substitute (5.8) into the right hand side of (5.10). Thus, for the proof of Prop. 5.5 we have to show that each continuous character α on $I_{c,n}^{\infty}(G)$ takes the form

$$\alpha(f) = \int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t}\, dt,\ f \in I_{c,n}^{\infty}(G)\,. \tag{5.11}$$

for some $\lambda \in \mathbf{C}$. In §5.5 we will prove:

THEOREM 5.6. *Let* $G = SU(1,1)$, $n \in \frac{1}{2}\mathbf{Z}$. *The mapping* $f \to F_f^n$ *is a topological algebra isomorphism from* $I_{c,n}^{\infty}(G)$ *onto* $\mathscr{D}_{\mathrm{even}}(\mathbf{R})$, *the algebra of even* C^{∞}*-functions with compact support on* $\mathbf{R}$.

Thus, in view of (5.11) we are left to prove:

PROPOSITION 5.7. *The continuous characters on* $\mathscr{D}_{\mathrm{even}}(\mathbf{R})$ *have the form*

$$h \to \int_{-\infty}^{\infty} h(t) e^{-\lambda t}\, dt$$

for some $\lambda \in \mathbf{C}$.

5.5. COMPLETION OF THE PROOF OF THE MAIN THEOREM

By the discussion in §5.4 we reduced the proof of Theorem 5.4 to the task of proving Theorem 5.6 and Prop. 5.7. Theorem 5.6 was partly proved in Prop. 5.3. It is left to prove that $f \to F_f^n$ is injective on $I_{c,n}^{\infty}(G)$ with image $\mathscr{D}_{\mathrm{even}}(\mathbf{R})$ and that the inverse mapping is continuous. In order to establish this we identify both $I_{c,n}^{\infty}(G)$ and $\mathscr{D}_{\mathrm{even}}(\mathbf{R})$, considered as topological vector spaces, with $\mathscr{D}([1,\infty))$ and we rewrite $f \to F_f^n$ as a mapping from $\mathscr{D}([1,\infty))$ onto itself. This mapping turns out to be a known integral transformation, for which an inverse transformation can be explicitly given. First note:

LEMMA 5.8. *The formula*

$$f(x) = g(x^2) \tag{5.12}$$

defines an isomorphism of topological vector spaces $f \to g$ *from* $\mathscr{D}_{\mathrm{even}}(\mathbf{R})$ *onto* $\mathscr{D}([0,\infty))$.