5.5. COMPLETION OF THE PROOF OF THE MAIN THEOREM

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **15.09.2024**

Now substitute (5.8) into the right hand side of (5.10). Thus, for the proof of Prop. 5.5 we have to show that each continuous character α on $I_{c,n}^{\infty}(G)$ takes the form

$$\alpha(f) = \int_{-\infty}^{\infty} F_f^n(t)e^{-\lambda t}\,dt,\ f \in I_{c,n}^{\infty}(G)\,. \tag{5.11}$$

for some $\lambda \in \mathbf{C}$. In §5.5 we will prove:

THEOREM 5.6. *Let* $G = SU(1,1)$, $n \in \frac{1}{2}\mathbf{Z}$. *The mapping* $f \to F_f^n$ *is a topological algebra isomorphism from* $I_{c,n}^{\infty}(G)$ *onto* $\mathscr{D}_{\text{even}}(\mathbf{R})$, *the algebra of even* C^{∞}*-functions with compact support on* $\mathbf{R}$.

Thus, in view of (5.11) we are left to prove:

PROPOSITION 5.7. *The continuous characters on* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *have the form*

$$h \to \int_{-\infty}^{\infty} h(t)e^{-\lambda t}\,dt$$

for some $\lambda \in \mathbf{C}$.

5.5. Completion of the proof of the main theorem

By the discussion in §5.4 we reduced the proof of Theorem 5.4 to the task of proving Theorem 5.6 and Prop. 5.7. Theorem 5.6 was partly proved in Prop. 5.3. It is left to prove that $f \to F_f^n$ is injective on $I_{c,n}^{\infty}(G)$ with image $\mathscr{D}_{\text{even}}(\mathbf{R})$ and that the inverse mapping is continuous. In order to establish this we identify both $I_{c,n}^{\infty}(G)$ and $\mathscr{D}_{\text{even}}(\mathbf{R})$, considered as topological vector spaces, with $\mathscr{D}([1,\infty))$ and we rewrite $f \to F_f^n$ as a mapping from $\mathscr{D}([1,\infty))$ onto itself. This mapping turns out to be a known integral transformation, for which an inverse transformation can be explicitly given. First note:

LEMMA 5.8. *The formula*

$$f(x) = g(x^2) \tag{5.12}$$

defines an isomorphism of topological vector spaces $f \to g$ *from* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *onto* $\mathscr{D}([0,\infty))$.

Proof. Clearly, if $g \in \mathscr{D}([0, \infty))$ then $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$ and the mapping $g \to f$ is continuous. Conversely, let $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$ and let g be defined by (5.12). By complete induction with respect to n we prove: $g^{(n)}(0)$ exists and there is a function $f_n \in \mathscr{D}_{\text{even}}(\mathbf{R})$ such that

$$f_n(x) = g^{(n)}(x^2),\ x \in \mathbf{R}\,,$$

and $f \to f_n : \mathscr{D}_{\text{even}}(\mathbf{R}) \to \mathscr{D}_{\text{even}}(\mathbf{R})$ is continuous. Indeed, suppose this is proved up to $n - 1$. Then

$$2x(g^{(n-1)})'(x^2) = f'_{n-1}(x) = \int_0^x f''_{n-1}(y)dy\,,$$

so

$$g^{(n)}(x^2) = \tfrac{1}{2}\int_0^1 f''_{n-1}(tx)dt =: f_n(x)\,. \qquad \square$$

For $f \in I^{\infty}_{c,n}(G)$ define

$$\text{(5.13)} \qquad \tilde{f}(x) := f\left(\begin{pmatrix} x & (x^2-1)^{\frac{1}{2}} \\ (x^2-1)^{\frac{1}{2}} & x \end{pmatrix}\right),\ x \in [1, \infty)\,.$$

For $f \in I^{\infty}_{c,n}(G)$ define

$$\text{(5.14)} \qquad \tilde{h}(ch\tfrac{1}{2}t) := h(t),\ t \in \mathbf{R}\,.$$

LEMMA 5.9. *The mapping* $f \to \tilde{f}$ *defined by* (5.13) *is an isomorphism of topological vector spaces from* $I^{\infty}_{c,n}(G)$ *onto* $\mathscr{D}([1, \infty))$. *The mapping* $h \to \tilde{h}$ *defined by* (5.35) *is an isomorphism of topological vector spaces from* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *onto* $\mathscr{D}([1, \infty))$.

Proof. The second statement follows from Lemma 5.8. For the proof of the first statement introduce global real analytic coordinates on G by the mapping

$$(z, \phi) \to \begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}$$

from $C \times (\mathbf{R}/4\pi\mathbf{Z})$ onto G. If $g \in \mathscr{D}([1, \infty))$ and

$$f\left(\begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}\right) := e^{in\phi}g((1+|z|^2)^{\frac{1}{2}})$$

then $f \in I^{\infty}_{c,n}(G)$, $\tilde{f} = g$ and the mapping $g \to f$ is continuous. Conversely, if $f \in I^{\infty}_{c,n}(G)$ then f, as a function of z and ϕ, is radial in z, so the function

$$z \to f\begin{pmatrix} (1+z^2)^{\frac{1}{2}} & z \\ z & (1+z^2)^{\frac{1}{2}} \end{pmatrix}, z \in \mathbf{R},$$

belongs to $\mathcal{D}_{\text{even}}(\mathbf{R})$. Now make the transformation $z = (x^2-1)^{\frac{1}{2}}$ and apply Lemma 5.8. It follows that $\tilde{f} \in \mathcal{D}([1, \infty))$ and that the mapping $f \to \tilde{f}$ is continuous. □

Define the Chebyshev polynomial $T_n(x)$ by

$$T_n(\cos\theta) := \cos n\theta. \tag{5.15}$$

It follows from (5.7) that, for $f \in I^{\infty}_{c,n}(G)$:

$$F^n_f(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f\left(\begin{pmatrix} ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t} & * \\ * & * \end{pmatrix}\right) dz$$

$$= e^{\frac{1}{2}t} \int_{-\infty}^{\infty} \tilde{f}(|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|)\left(\frac{ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t}}{|ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t}|}\right)^{2n} dz$$

$$= e^{\frac{1}{2}t} \int_{0}^{\infty} \tilde{f}(|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|) T_{2|n|}\left(\frac{ch\frac{1}{2}t}{|ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t}|}\right) dz,$$

so

$$F^n_f(t) = 2 \int_{ch\frac{1}{2}t}^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1} ch\tfrac{1}{2}t)(y^2 - ch^2\tfrac{1}{2}t)^{-\frac{1}{2}} y\,dy.$$

This formula shows that F^n_f is even on $\mathbf{R}$, so $F^n_f \in \mathcal{D}_{\text{even}}(\mathbf{R})$. Now, by (5.14):

$$\tilde{F}^n_f(x) = 2 \int_{x}^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1}x)(y^2 - x^2)^{-\frac{1}{2}} y\,dy, \quad x \in [1, \infty]. \tag{5.16}$$

For $n = 0$, (5.16) takes the form

$$\tilde{F}^0_f(x) = 2 \int_{x}^{\infty} \tilde{f}(y)(y^2 - x^2)^{-\frac{1}{2}} y\,dy;$$

The problem of inverting this just means to solve the Abel integral equation, as was pointed out by GODEMENT [20]. Indeed, we get

$$\tilde{f}(y) = -\pi^{-1} \int_y^\infty \frac{d}{dx} \tilde{F}_f^0(x)(x^2-y^2)^{-\frac{1}{2}} dx .$$

For general n, we can use an inversion formula obtained by DEANS [7, (30)], see also MATSUSHITA [30, §2.3] and KOORNWINDER [27, §5.9]:

THEOREM 5.10. *For* $m = 0, 1, 2, \ldots, g \in \mathscr{D}([1, \infty)), x \in [1, \infty)$ *define*

$$(5.17) \qquad (A_m g)(x) := 2 \int_x^\infty g(y) T_m (y^2-x^2)^{-\frac{1}{2}} y dy ,$$

$$(5.18) \qquad (B_m g)(x) := -\pi^{-1} \int_x^\infty g'(y) T_m(x^{-1}y)(y^2-x^2)^{-\frac{1}{2}} dy .$$

Then A_m *and* B_m *map* $\mathscr{D}([1, \infty))$ *into itself and* $A_m B_m = id, B_m A_m = id.$

This theorem shows that $f \to F_f^n$ is a linear bijection from $I_{c,n}^\infty(G)$ onto $\mathscr{D}_{\text{even}}(\mathbf{R})$. Finally in order to prove the continuity of the inverse mapping, we show that B_m is continuous. Just expand $T_m(x^{-1}y)$ as a polynomial and use that

$$\left(x^{-1}\frac{d}{dy}\right)^p \int_x^\infty h(y)(y^2-x^2)^{-\frac{1}{2}} y dy$$

$$= \int_x^\infty \left(y^{-1}\frac{d}{dy}\right)^p h(y)(y^2-x^2)^{-\frac{1}{2}} y dy$$

by the properties of the Weyl fractional integral transform (cf. [11, Ch. 13]). This completes the proof of Theorem 5.6.

Proof of proposition 5.7. Extend α to a continuous linear functional on $\mathscr{D}(\mathbf{R})$, for instance by putting $\alpha(f) = 0$ if f is odd. Choose $f_1 \in \mathscr{D}_{\text{even}}(\mathbf{R})$ such that $\alpha(f_1) \neq 0$. Let

$$(\lambda(y)f_1)(x) := f_1(x-y), \; x, y \in \mathbf{R} .$$

By the continuity and homomorphism property of α we have, for $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$:

$$\alpha(f_1)\alpha(f) = \alpha(f_1 * f) = \int_{-\infty}^{\infty} \alpha(\lambda(y)f_1)f(y)dy\,.$$

Hence

$$\alpha(f) = \int_{-\infty}^{\infty} f(y)\beta(y)dy,\ f \in \mathscr{D}_{\text{even}}(\mathbf{R})\,,$$

where

$$\beta(y) := \tfrac{1}{2}(\alpha(f_1))^{-1}(\alpha(\lambda(y)f_1) + \alpha(\lambda(-y)f_1))\,.$$

Then β is even and it is a continuous function by the continuity of α. It follows from the homomorphism property of α and from the fact that β is even, that

$$\beta(x)\beta(y) = \tfrac{1}{2}(\beta(x+y) + \beta(x-y))\,,$$

so $\beta(0) = 1$. This is d'Alembert's functional equation. By continuity, $Re\ \beta(x) > 0$ if $0 \leqslant x \leqslant x_0$ for some $x_0 > 0$. Then $\beta(x_0) = \cosh c$ for some complex $c = a + ib$ with $a \geqslant 0$, $-\tfrac{1}{2}\pi < b < \tfrac{1}{2}\pi$. Now, following the proof in ACZEL [1, 2.4.1] it can be shown [1]) that for all integer $n, m \geqslant 0$

$$\beta\left(\frac{n}{2^m}x_0\right) = \cosh\left(\frac{c}{x_0}\frac{n}{2^m}x_0\right).$$

So, by continuity and evenness of β:

$$\beta(x) = \cosh\left(\frac{c}{x_0}x\right) \text{ for all } x \in \mathbf{R}\,. \qquad \square$$

5.6. NOTES

5.6.1. Some other examples of Gelfand pairs $(G \times K, K^*)$ are provided by $G = SO_0(n, 1)$, $K = SO(n)$ and $G = SU(n, 1)$, $K = S(U(n) \times U(1))$, cf. BOERNER [4, Ch. VII, §12; Ch. V, §6], DIXMIER [8] or KOORNWINDER [27, Theorems 5.7, 5.8].

5.6.2. The main Theorem 5.4, which was first proved in the case of unitary representations by BARGMANN [2], is a special case of the *subrepresentation*

[1]) I thank H. van Haeringen for this reference.