II. Bounded Ultrapowers

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Here is a brief description of the simplest model which we construct later. Let $\{P_i\}$ be an effective enumeration of all primitive recursive partitions $P_i : [\mathbf{N}]^{e\,i} \to r_i$. It is an immediate consequence of Ramsey's Theorem that there exists a finite set X_k in $\mathbf{N}$, with $\# X_k \geqslant k$, min X_k, which is homogeneous for $P_1, \ldots, P_k$. Let $\{a_{k1}, \ldots, a_{kn_k}\}$ be an enumeration of X_k in increasing order. Let the functions $h_j : \mathbf{N} \to \mathbf{N}$ be defined by

$$h_j(k) = \begin{cases} a_{kj} & j \leqslant n_k \\ h_{j-1}(k)^2 & j > n_k . \end{cases}$$

Now let

$$\mathscr{F} = \{f \mid \exists j \, \forall i \, f(i) \leqslant h_j(i)\} .$$

Then, for any non-principal ultrafilter D, the restricted ultrapower $\mathscr{F}/D$ is a non-standard model of Peano arithmetic. If, in addition, we assume that X_k has been chosen so that a_{kn_k} is a minimum, then the above consequence of Ramsey's Theorem is false in this model.

II. Bounded Ultrapowers

In building the model we have endeavoured to motivate each stage of the construction. Since this is a modification of the ultraproduct construction it is natural to aim at reproducing (to a degree) the main properties of the full ultraproduct. The first property of the ultraproduct we mimic is the Łoš property that a formula is satisfied in the ultraproduct if and only if it is satisfied in a set of factors lying in the ultrafilter. Of course, we wish to have this true for only a limited set of formulas to avoid constructing a model elementarily equivalent to $\mathbf{N}$.

By a *limited formula* we mean one in which every quantifier occurs in bound form: $\forall\, x < z$ or $\exists\, x < z$.

If $f, g \in \mathbf{N}^I$, we write $f \leqslant g$ to mean $f(i) \leqslant g(i)$ for all $i \in I$. A natural constraint on our proposed set $\mathscr{F}$ is that it be closed under $\leqslant$, i.e. $f \leqslant g \in \mathscr{F}$ implies $f \in \mathscr{F}$. We call the restricted ultrapower $\mathscr{F}/D$ resulting from such an $\mathscr{F}$ a *bounded ultrapower*. This condition is sufficient to prove the Łoš property for limited formulas.

The formal language we use for Peano arithmetic has the constant 0 and two binary relation symbols $\sigma(x, y, z)$ and $\pi(x, y, z)$ (denoting $x+y = z$ and $x \cdot y = z$ in $\mathbf{N}$). By not having the functions $+$ and $\cdot$ in the language we avoid having to assume at the outset that $\mathscr{F}/D$ is closed under $+$ and $\cdot$.

THEOREM 1. *Assume* $\mathscr{F}/D$ *is a bounded ultrapower. Let* $\phi(x_1, \ldots, x_n)$ *be a limited formula and* $f_1, \ldots, f_n \in \mathscr{F}$. *Then* $\mathscr{F}/D \models \phi(f_1^*, \ldots, f_n^*)$ *if and only if* $\{ i \mid \mathbf{N} \models \phi(f_1(i), \ldots, f_n(i)\} \in D$.

Proof: We proceed by induction on the length of ϕ. For atomic formulas the equivalence is an immediate consequence of the definition of $\mathscr{F}/D$. That the equivalence is preserved under the logical connectives follows exactly as in the full ultraproduct case from the properties of the ultrafilter D.

Now assume that $\phi(x_1, \ldots, x_n)$ has the form $(\exists x_j < x_k)\psi(x_1, \ldots, x_n, x_j)$. Suppose that

$$s = \{i \mid \mathbf{N} \models (\exists x_j < f_k(i))\psi(f_1(i), \ldots, f_n(i), x_j)\} \in D\,.$$

Then for each $i \in s$, there exists in $\mathbf{N}$ an element $a_i < f_k(i)$ such that

$$\mathbf{N} \models \psi(f_1(i), \ldots, f_n(i), a_i)\,.$$

Define the functions $g : \mathbf{N} \to \mathbf{N}$ by

$$g(i) = \begin{cases} a_i & \text{for } i \in s \\ 0 & \text{for } i \notin s\,. \end{cases}$$

Since $g < f_k \in \mathscr{F}$, we have $g \in \mathscr{F}$. Now $\{ i \mid \mathbf{N} \models \psi(f_1(i), \ldots, f_n(i), g(i)\}$ $= s \in D$. By the inductive hypothesis

$$\begin{aligned} & \mathscr{F}/D \models \psi(f_1^*, \ldots, f_n^*, g^*) \\ \text{or} \quad & \mathscr{F}/D \models (\exists x_j < f_k^*)\psi(f_1^*, \ldots, f_n^*, x_j) \\ \text{i.e.} \quad & \mathscr{F}/D \models \phi(f_1^*, \ldots, f_n^*)\,. \end{aligned}$$

The other half of the equivalence is immediate.

We can extend this result a little further in one direction. The proof of the following consequence is obvious.

COROLLARY. *Let* $\phi(x_1, \ldots, x_n)$ *be a* $\sum_1^0$*-formula. Then*

$$\mathscr{F}/D \models \phi(f_1^*, \ldots, f_n^*) \text{ implies } \{ i \mid \mathbf{N} \models \phi(f_1(i), \ldots, f_n(i)\} \in D\,.$$

The second property of ultraproducts we copy is the saturation property. The ω_1-saturation property of a structure $\mathscr{A}$ is usually formulated as follows. Let $\{\phi_j(z)\}$ be a countable sequence of formulas with coefficients in $\mathscr{A}$ and the indicated free variable z. Then $\mathscr{A} \models \exists z \bigwedge_{j \leq n} \phi_j(z)$ for every n implies $\mathscr{A} \models \exists z \bigwedge_{j=i}^{\infty} \phi_j(z)$.

An immediate consequence of the saturation property is the following apparently stronger statement: Let $\{ \phi_j (z_1, ..., z_{n_j}) \}$ be a countable sequence of formulas with coefficients in $\mathscr{A}$ and with the indicated free variables. Then

$$\mathscr{A} \models \exists z_1 \ldots \exists z_{n_k} \bigwedge_{j \leq k} \phi_j (z_1, \ldots, z_{n_j})$$

for every k implies that

$$\mathscr{A} \models \exists z_1 \ldots \exists z_n \ldots \bigwedge_{j=1}^{\infty} \phi_j (z_1, \ldots, z_{n_j}) \; .$$

It is this form of the saturation property which we shall adapt in constructing the model $\mathscr{F}/D$. We shall require the property only for a fixed sequence $\{ \phi_j \}$ of limited formulas which we shall specify later in the construction of the model. We shall in addition find it useful to add a condition relating the free variable z_1 with k and n_k in the form of a limited formula $\phi (k, n_k, z_1)$. We could here replace z_1 by a finite sequence of the free variables but we shall not need this added generality.

Let $\phi (x, y, z_1)$ and $\phi_j (z_1, ..., z_{n_j})$, $j = 1, 2, 3, ...$, be limited formulas with the indicated free variables. We assume that n_j increases with j. Suppose that for each k

$$\mathbf{N} \models \exists z_1 \ldots \exists z_{n_k} (\phi (k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j (z_1, \ldots, z_{n_j}))$$

Given k, let $a_{k1}, ..., a_{kn_k} \in \mathbf{N}$ be such that

$$\mathbf{N} \models \phi (k, n_k, a_{k1}) \wedge \bigwedge_{j \leq k} \phi_j (a_{k1}, \ldots, a_{kn_j}) \, .$$

Define the functions h_j by

$$h_o (k) = n_k \quad \text{for all } k \, ,$$

and for $j > 0$,

$$h_j (k) = \begin{cases} a_{kj} & \text{for } n_k \geqslant j \\ \text{arbitrary} & \text{for } n_k < j \, . \end{cases}$$

THEOREM 2. *Let* $\mathscr{F} \subseteq \mathbf{N}^{\mathbf{N}}$ *contain the functions* $h_j, j = 0, 1, 2, ...,$ *and* **1** *(the identity function) and be closed under* $<$. *Then*

$$\mathscr{F}/D \models \exists x \, \exists y \, \exists z_1 \ldots \exists z_n \ldots (\phi (x, y, z_j) \wedge \bigwedge_{j=1}^{\infty} \phi_j (z_1, \ldots, z_{n_j}))$$

Proof: Since the formula ϕ_m occurs as a conjunct in $\bigwedge_{j \leq n} \phi_j$ for $n_k \geqslant m$, we have $\mathbf{N} \models \phi_m (a_{k1}, \ldots, a_{km})$. i.e. $\mathbf{N} \models \phi_m (h_1 (k), ..., h_m (k))$. Thus, by

Theorem 1

$$\mathscr{F}/D \models \phi_m(h_1^*, \dots, h_m^*) .$$

Again, for every k

$$\mathbf{N} \models \phi(k, n_k, a_{k1}), \text{ so that}$$

by Theorem 1

$$\mathscr{F}/D \models \phi(\mathbf{1}^*, h_0^*, h_1^*) ,$$

proving the theorem.

We shall in the sequel be taking for $\mathscr{F}$ the smallest set of functions closed under $<$ and containing $\mathbf{1}$ and all the h_j's. In other words we let

$$\mathscr{F} = \{f \mid f \leqslant \mathbf{1} \quad \text{or} \quad \exists j\, f \leqslant h_j\} .$$

As an example of the use of the Saturation Theorem 2 we show how we can ensure that $\mathscr{F}$ is closed under $+$ and $\cdot$. Since

$$f, g < h_j \quad \text{implies} \quad f + g, f \cdot g < h_j^2$$

it clearly suffices to assume that $h_{j-1}^2 < h_j$. Thus, if we assume that the condition $z_{j-1}^2 < z_j$ occurs in the formula ϕ_j, then, adding $h_j(k) = h_{j-1}^2(k)$ for $j > n_k$ to our definition of h_j, we have that $h_{j-1}^2 < h_j$, so that $\mathscr{F}$ is closed under $+$ and $\cdot$. We shall call this the Closure Condition on $\{h_j\}$.

Again, we can guarantee that $\mathbf{1} \in \mathscr{F}$ by assuming that $\phi(k, n, z_1)$ includes the condition $z_1 > k$.

Up to this point in our construction of $\mathscr{F}/D$ there is no guarantee that the difficulty with the full ultrapower has been obviated. It may happen that $\mathscr{F}/D \equiv N$, so that $\mathscr{F}/D$ cannot be used for independence results. To obtain a true arithmetical statement which is false in $\mathscr{F}/D$ we now add the condition that the sequence $\{\phi_j\}$ of limited formulas is a recursively enumerable set. It follows immediately that the sequence $\{\bigwedge_{j \leq n} \phi_j\}$ is also a recursively enumerable set. Since the satisfaction relation for limited formulas in $\mathbf{N}$ is a primitive recursive relation there is a $\sum_1^0$-formula $\gamma(x, y, z)$ such that

$$\mathbf{N} \models \gamma(k, n_k, z) \leftrightarrow \phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \dots, z_{n_j}) .$$

Here $\gamma(k, n_k, z)$ holds if and only if z is a code number of a sequence of length n_k such that $\phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \dots, z_{n_j})$ holds, where $z_i = \beta(i, z)$ and β is the Gödel β-function (as given e.g. in Shoenfield [8] §6.4.)

Suppose that, as in the hypothesis of Theorem 2, we assume that for every k

$$\mathbf{N} \models \exists z_1 \dots \exists z_{n_k} [\phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \dots, z_{n_j})]$$

We now construct the functions $\{ h_j \}$ of Theorem 2 with greater care. For each k, we choose the sequence $a_{k1}, \dots, a_{kn_k}$ to be the *least* sequence satisfying

$$\phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \dots, z_{n_j})$$

in $\mathbf{N}$. The precise measure of what we mean by least is not critical, but we shall take it to mean that the largest element of $a_{k1}, \dots, a_{kn_k}$ is a minimum for all possible choices of $a_{k1}, \dots, a_{kn_k}$ satisfying the above formula. We shall henceforth assume by appropriate re-labeling that $a_{k1} < a_{k2} < \dots < a_{kn_k}$, so that a_{kn_k} is the minimal element. We now claim that

$$\mathscr{F}/D \models \neg \forall k \exists n \exists z_1 \dots \exists z_{n_k} [\phi(k, n, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \dots, z_{n_j})]$$

or, more precisely,

$$\mathscr{F}/D \models \neg \forall k \exists n \exists z \, \gamma(k, n, z) .$$

Since $\mathbf{N} \models \forall k \exists z \, \gamma(k, n_k, z)$ we have obtained a true arithmetical statement which is false in $\mathscr{F}/D$. Note that it follows from Theorem 2 that for all $k \in \mathbf{N}$

$$\mathscr{F}/D \models \exists n \exists z \, \gamma(k, n, z) .$$

THEOREM 3. $\mathscr{F}/D \models \neg \forall k \exists n \exists z \, \gamma(k, n, z)$.

Proof: Assume that on the contrary

$$\mathscr{F}/D \models \forall k \exists n \exists z \, \gamma(k, n, z) .$$

Choose $k = \mathbf{1}^*$. Then there exist $r, g \in \mathscr{F}$ such that

$$\mathscr{F}/D \models \gamma(\mathbf{1}^*, r^*, g^*) .$$

By the Corollary to Theorem 1 we have for an infinite set t of k's (lying in the ultrafilter D)

$$\mathbf{N} \models \gamma(k, r(k), g(k)) \tag{1}$$

Now since $\beta(r, g) \leqslant r$ [1] $\in \mathscr{F}$, there exists an i such that

$$\beta(r, g) < h_i .$$

[1]) See Shoenfield [8] § 6.4.

Choose $k > i$ lying in t. Then $h_i < h_k$, so that

$$g_{r(k)}(k) = \beta(r(k), g(k)) < h_k(k) = a_{kn_k}.$$

But by (1) the sequence

$$g_1(k), \ldots, g_{r(k)}(k)$$

satisfies the above formula. This contradicts the minimality of a_{kn_k}.

III. Peano arithmetic and the Stability Condition

Theorem 1 suffices to construct a non-standard model of a theory of arithmetic in which all the axioms are expressed by limited formulas. The induction axioms of Peano arithmetic however involve arbitrary elementary formulas. To deal with this problem we shall associate with each formula $\phi(y)$ of arithmetic a limited formula $\hat{\phi}(y; z)$ [1] called the *limited associate* of $\phi(y)$.

We assume that $\phi(y)$ has been reduced to prenex normal form. To obtain the formula $\hat{\phi}(y; z)$ we replace each quantifier Qx_i in $\phi(y)$ by the bounded quantifier $Qx_i < z_i$. The bounding variables z_k are to be distinct from the variables occurring in $\phi(y)$ and also distinct from each other.

Although Theorem 1 allows us to prove the validity of limited associates of the Peano axioms in the model $\mathscr{F}/D$, we need a provision for inferring from this the validity of the Peano axioms themselves in $\mathscr{F}/D$.

To obtain the desired result it would suffice to show that for some suitable vector h in $\mathscr{F}$, $\mathscr{F}/D \models \hat{\phi}(y; h^*)$ implies $\mathscr{F}/D \models \phi(y)$. However, if we consider the case where $\phi(y)$ is $(\forall x)(y \neq x)$, we find $\mathscr{F}/D \models \hat{\phi}(h^*, h^*)$ but $\mathscr{F}/D \models \neg \phi(h^*)$. This example shows that we must restrict the range of y, i.e. require that for all $f < h$, $\mathscr{F}/D \models \hat{\phi}(f^*, h^*)$ implies $\mathscr{F}/D \models \phi(f^*)$. To prove $\mathscr{F}/D \models \phi(f^*)$ for all f in $\mathscr{F}$ it thus suffices to construct an increasing sequence $\{h_i\}$ in $\mathscr{F}$, cofinal in $\mathscr{F}$, such that for all i, j with $i < j$ [2] and all $f \in \mathscr{F}$ with $f < h_i$,

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[1]) Here and later y and z denote vectors of variables.

[2]) Here and later j denotes a vector $\langle j_1, \ldots, j_n \rangle$, $i < j$ means $i < \min j$, and h_j denotes the vector $\langle h_{j_1}, \ldots, h_{j_n} \rangle$.