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ON BOOLEAN ALGEBRAS WITH DISTINGUISHED SUBALGEBRAS *

by Sabine KOPPELBERG

In this paper, let $\mathcal{L} = \{+, \cdot, -, 0, 1, U\}$ be the language of Boolean algebras (BA 's) with an additional unary predicate \mathcal{U} . Rubin has proved in [6] that the theory in \mathcal{L} of Boolean algebras with a distinguished subalgebra (given by the interpretation of U) is undecidable. The main result of this paper is the solution of a problem stated in [6]: let \mathbf{K} be the class of \mathcal{L} -structures $\mathcal{M} = (B, +, \cdot, -, 0, 1, A)$ where (B, \dots) is a complete BA (cBA), A is a complete subalgebra and the inclusion map from A to B is complete; we show that $\text{Th}(\mathbf{K})$, the set of first-order \mathcal{L} -sentences which are true in every structure in \mathbf{K} , is decidable. We shall abbreviate BA 's (B, \dots) by their underlying set B .

The first idea to do this is to describe explicitly all completions of $\text{Th}(\mathbf{K})$. One could then try to prove the decidability of $\text{Th}(\mathbf{K})$ by Theorem 2 in [5]. A well-known example for a decidability proof in this style is given by the theory of BA 's; the main task, to list all completions of this theory, was achieved by Tarski, see Theorem 5.5.10 in [1]. Before describing the complete first-order theory of a structure $\mathcal{M} = (B, A)$ in \mathbf{K} , one has to get some idea how B "lies above A " and which details of the structure of an extension (B, A) of BA 's can be expressed in first-order logic. Now B can be represented by the set of global sections of a sheaf of BA 's over the Stone space X of A . Although the possibility of this representation is probably well-known to experts and although it is very easily established, it seems to give just the right intuition as to what are the relevant facts about the extension (B, A) . We thus get an idea how to obtain a recursive set T of \mathcal{L} -sentences which looks rather natural and holds in every structure \mathcal{M} of \mathbf{K} .

It turns out that Comer's Feferman-Vaught-theorem on sheaves over Boolean spaces applies to the models of T . This establishes rather easily that a first-order sentence is in $\text{Th}(\mathbf{K})$ if and only if it is provable from T

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and that $\text{Th}(\mathbf{K})$ is decidable. It is then possible to describe the completions of T (which, however, was not necessary in the decidability proof).

As another example for the usefulness of sheaf representations of BA extensions (B, A) , we consider the special case where B is finitely generated over A and we describe the action of a single automorphism of B leaving A pointwise fixed. This was motivated by Monk's paper [4] where the Galois group $\text{Aut}_A B$ is studied in detail for a simple extension B of A and attempts are made towards finite extensions. The possibility of describing by a sheaf representation those extensions (S, R) of commutative rings for which the usual Galois correspondence can be established is, however, not new- see [8].

In section 1 of this paper, we give a sketch of the sheaf representation of a BA extension (B, A) . We prove that the sheaf is Hausdorff iff A is relatively complete in B , which means that for $b \in B$, there is a largest $a \in A$ such that $a \leq b$.

In section 2, we provide a method to construct all automorphisms of B over A if B is a finite extension of A (2.4). We illustrate this method by computing the Galois group of B over A if A is relatively complete in B (2.6) and by proving in 2.7 that A is relatively complete in B iff there is a single automorphism of B over A moving every element of $B \setminus A$. This means that the finite extensions (B, A) where A is relatively complete in B are just the extensions called weakly Galois in [8].

Section 3 contains part of the machinery needed for the main result of the paper: if $(B, A) \in \mathbf{K}$, $\varphi(x_1 \dots x_n)$ is an \mathcal{L} -formula and $b_1, \dots, b_n \in B$, we prove that $\|\varphi[b_1 \dots b_n]\|$, the set of points p in the Stone space X of A such that φ is satisfied by $b_1(p), \dots, b_n(p)$ in the stalk B_p over p , is a clopen subset of X . This enables us to apply the Feferman-Vaught theorem in Comer's version to our sheaf. More precisely, we show that there is an effective procedure assigning a formula $s_\varphi(yx_1 \dots x_n)$ to $\varphi(x_1 \dots x_n)$ such that the element a of A corresponding to $\|\varphi[b_1 \dots b_n]\|$ is the only element of A satisfying $s_\varphi(ab_1 \dots b_n)$ in (B, A) . We then define the theory T in \mathcal{L} and show that each \mathcal{M} in \mathbf{K} is a model of T .

Finally in section 4, we prove that the theorems of T are just the sentences in $\text{Th}(\mathbf{K})$ and that $\text{Th}(\mathbf{K})$ is decidable. We characterize elementary equivalence of T -models, give a list of all completions of T and prove that each of these completions has a model in \mathbf{K} .

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1. THE SHEAF REPRESENTATION OF BOOLEAN ALGEBRA EXTENSIONS

Let \mathcal{L} be any language for first-order predicate logic. Suppose X is a non-empty set and for every $p \in X$ we have an \mathcal{L} -structure $\mathcal{B}_p = (B_p, \dots)$; put $S = \bigcup_{p \in X} B_p$. Suppose $\varphi(x_1 \dots x_n)$ is an \mathcal{L} -formula, $u \subseteq X$ and $f_1, \dots, f_n : u \rightarrow S$ are such that $f_i(p) \in B_p$ for $1 \leq i \leq n$ and $p \in u$. Then let

$$\| \varphi [f_1 \dots f_n] \| = \{ p \in u \mid \mathcal{B}_p \models \varphi [f_1(p) \dots f_n(p)] \} .$$

We may think of $\| \varphi [f_1 \dots f_n] \| \subseteq X$ as being a (Boolean) truth value of $\varphi [f_1 \dots f_n]$ in the power set of X .

A sheaf of \mathcal{L} -structures is a sequence

$$\mathcal{S} = (S, \pi, X, \mu)$$

such that a) S and X are topological spaces and $\pi : S \rightarrow X$ is a continuous open local homeomorphism from S onto X , b) μ is a function assigning to each $p \in X$ an \mathcal{L} -structure $\mathcal{B}_p = (B_p, \dots)$ such that the B_p are pairwise disjoint, $S = \bigcup_{p \in X} B_p$ and $\pi(s) = p$ iff $s \in B_p$, c) for every open subset u of X and continuous $f_1, \dots, f_n : u \rightarrow S$ satisfying $f_i(p) \in B_p$ for $p \in u$ and every atomic \mathcal{L} -formula $\varphi(x_1 \dots x_n)$, $\| \varphi [f_1 \dots f_n] \|$ is an open subset of u .

The \mathcal{L} -structure \mathcal{B}_p is called the stalk of \mathcal{S} over p . — Let, if \mathcal{S} is a sheaf of \mathcal{L} -structures, $\Gamma(\mathcal{S})$ be the set of all continuous functions $f : X \rightarrow S$ satisfying $f(p) \in B_p$ for $p \in X$ (the set of “global sections” of \mathcal{S}). So $\Gamma(\mathcal{S})$ is, if non-empty, (the underlying set of) a substructure of the product structure $\prod_{p \in X} \mathcal{B}_p$, hence an \mathcal{L} -structure.

For the rest of the paper, let $\mathcal{L} = \{ +, \cdot, -, 0, 1, U \}$ where U is a unary predicate. We indicate how, for a given BA extension (B, A) , B may be represented by $\Gamma(\mathcal{S})$ where \mathcal{S} is a sheaf of \mathcal{L} -structures over a Boolean space. We omit most of the proofs which are easy and entirely analogous to well-known representation theorems for lattices over Boolean spaces. Let X be the Stone space of A , i.e. the set of all ultrafilters of A with the usual topology. For $p \in X$, let $\langle p \rangle$ be the filter of B generated by p . Let $\pi_p : B \rightarrow B / \langle p \rangle = B_p$ be the canonical epimorphism. So B_p is a BA with at least two elements. For $p, q \in X$ and $p \neq q$, B_p and B_q are disjoint. Let $S = \bigcup_{p \in X} B_p$ and $\pi : S \rightarrow X$ be defined as stated in b) above. Let, for $p \in X$, $\mu(p)$ be the \mathcal{L} -structure $(B_p, \dots, \{0, 1\})$. For $u \subseteq X$ open and $b \in B$, let $M_{ub} = \{ \pi_p(b) \mid p \in u \}$. The set of these M_{ub} constitutes a base

for a topology of S , and this makes $\mathcal{S} = (S, \pi, X, \mu)$ a sheaf of \mathcal{L} -structures. Furthermore, for $b \in B$, $\sigma_b : X \rightarrow S$ defined by $\sigma_b(p) = \pi_p(b)$ is a global section of \mathcal{S} and

$$\left. \begin{array}{l} \sigma : B \rightarrow \Gamma(\mathcal{S}) \\ b \mapsto \sigma_b \end{array} \right\}$$

is an isomorphism from B onto $\Gamma(\mathcal{S})$. We shall now identify B with $\Gamma(\mathcal{S})$; so every $b \in B$ is a function from X to S . This identifies A with those $b \in B$ such that for every $p \in X$ $b(p) = 0$ or $b(p) = 1$, i.e. with those $b \in B$ satisfying $\|U(b)\| = X$. Let C be the BA of clopen subsets of X and $e(c)$ the characteristic function of c for $c \in C$. Thus e is an isomorphism from C onto $A \subseteq B$.

In the rest of this section, we show that the property of being a Hausdorff sheaf for \mathcal{S} is equivalent to a property of the extension (B, A) which reflects, in a way which is first-order expressible in \mathcal{L} , completeness of the embedding of A into B . Recall that, for a sheaf \mathcal{S} over a Boolean space X , S is a T_2 -space iff, for any $f, g \in \Gamma(\mathcal{S})$, $\|f = g\|$ is a clopen subset of X ; \mathcal{S} is then said to be a Hausdorff sheaf. Call A relatively complete in B if, for every $b \in B$, there is a largest element $a \in A$ such that $a \leq b$, equivalently: for $b \in B$, there is a largest $a \in A$ such that $a \cdot b = 0$ or: for $b \in B$, there is a smallest $a \in A$ such that $b \leq a$.

1.1. PROPOSITION. \mathcal{S} is a Hausdorff sheaf iff A is relatively complete in B .

Proof. Suppose \mathcal{S} is Hausdorff and $b \in B$. Let $d \in B$ such that $d(p) = 0$ for every $p \in X$, let $c = \|b = d\|$ and $a = e(c)$. Then a is the largest element of A satisfying $a \cdot b = 0$.

Conversely, let A be relatively complete in B and suppose $f, g \in B$. Let a be the largest element of A such that $a \leq f \cdot g + -f \cdot -g$. Let $c \in C$ such that $a = e(c)$. Then $\|f = g\| = c$ is a clopen subset of X .

1.2. REMARK. Let A be relatively complete in B . Then the inclusion map from A to B is a complete homomorphism.

Proof. Suppose M is a subset of A having a supremum a in A . We show that a is the supremum of M in B . Clearly, a is an upper bound for M in B . Suppose that b is another upper bound for M in B . Let $\alpha \in A$ be the largest element of A such that $\alpha \leq b$. For every $m \in M \subseteq A$, we have $m \leq b$, hence $m \leq \alpha$ and $a \leq \alpha \leq b$.

The following facts are trivial:

1.3. REMARK. a) Let A and the inclusion map from A to B be complete. Then A is relatively complete in B .

b) Suppose A is relatively complete in B and B is complete. Then A is complete.

2. RELATIVE AUTOMORPHISMS OF FINITE EXTENSIONS

We first give an internal description of a finite extension (B, A) where $B = A(u_1 \dots u_n)$ and $n \in \omega$. We shall always assume that u_1, \dots, u_n are the atoms of the subalgebra of B generated by u_1, \dots, u_n ; i.e. that they are non-zero, pairwise disjoint and $u_1 + \dots + u_n = 1$. Let $I_r = \{a \in A \mid a \cdot u_r = 0\}$ for $1 \leq r \leq n$. Clearly, each I_r is a proper ideal of A and $I_1 \cap \dots \cap I_n = \{0\}$. The family $(I_r \mid 1 \leq r \leq n)$ completely characterizes the extension (B, A) :

2.1. REMARK. Suppose $C = A(v_1 \dots v_n)$ is a finite extension of A where v_1, \dots, v_n are pairwise disjoint and $1 = v_1 + \dots + v_n$. Let $B = A(u_1 \dots u_n)$ be as above. There is an isomorphism g from B onto C satisfying $g(a) = a$ for $a \in A$ and $g(u_r) = v_r$ iff, for each r , $\{a \in A \mid a \cdot v_r = 0\} = I_r$.

Proof. By Theorem 12.4 in [7].

2.2. REMARK. A is relatively complete in $B = A(u_1 \dots u_n)$ iff, for each r , I_r is a principal ideal.

Proof. The only-if part follows by the definition of relative completeness. Now suppose $\alpha_r \in A$ generates I_r ; let $b \in B$ and $I = \{a \in A \mid a \cdot b = 0\}$. There are $a_1, \dots, a_n \in A$ such that $b = a_1 \cdot u_1 + \dots + a_n \cdot u_n$. It follows that I is the principal ideal generated by $\alpha = (-a_1 + \alpha_1) \cdot \dots \cdot (-a_n + \alpha_n)$.

Conversely, given any family $(I_r \mid 1 \leq r \leq n)$ of proper ideals in A satisfying $I_1 \cap \dots \cap I_n = \{0\}$, there is an extension $A(u_1 \dots u_n)$ of A such that $I_r = \{a \in A \mid a \cdot u_r = 0\}$: let $D = A(x_1 \dots x_n)$ be the free product of A and a finite BA with atoms x_1, \dots, x_n . Let

$$K = \{i_1 \cdot x_1 + \dots + i_n \cdot x_n \mid i_1 \in I_1, \dots, i_n \in I_n\}.$$

K is an ideal of D ; the canonical epimorphism π from D onto $B = D/K$ is one-one on A , and for $a \in A$, $\pi(a) \cdot u_r = 0$ iff $a \in I_r$ where $u_r = \pi(x_r)$. Now identify A with the subalgebra $\pi(A)$ of B .

For the rest of this section we think, as in section 1, of B as being the set of global sections of a sheaf $\mathcal{S} = (S, \pi, X, \mu)$ of Boolean algebras over a

Boolean space X ; we use the abbreviations of section 1. For $p \in X$, $B_p = \{b(p) \mid b \in B\}$. Since $b(p) \in \{0, 1\}$ for $b \in A$ and $B = A(u_1 \dots u_n)$, B_p is a finite BA with atoms $\{u_r(p) \mid 1 \leq r \leq n\} \setminus \{0\}$.

Let $G = \text{Aut}_A B$ be the group of those automorphisms of B leaving A pointwise fixed, i.e. G is the Galois group of B over A . Suppose $g \in G$ and $p \in X$. Since $g(a) = a$ for $a \in A$, g induces an automorphism of B_p which, in turn, is induced by a permutation of the (at most n) atoms of B_p . This gives rise to the following definitions (S_n is the group of permutations of $\{1, \dots, n\}$).

Let $p \in X$. For $1 \leq r, l \leq n$, say $u_r \sim u_l$ at p if there is a neighbourhood u of p such that, for $q \in u$, $u_r(q) = 0$ iff $u_l(q) = 0$. $\pi \in S_n$ is said to be compatible with p if $u_r \sim u_{\pi(r)}$ at p for $1 \leq r \leq n$. $g \in G$ is said to be induced by π at p if $g(u_r)(p) = u_{\pi(r)}(p)$ for $1 \leq r \leq n$. Note that, if one of these definitions holds (for fixed $u_r, u_l, \pi \in S_n, g \in G$) for some $p \in X$, then it holds (for the same $u_r, u_l, \pi \in S_n, g \in G$) for every q in some neighbourhood of p . And $u_r \sim u_l$ at p means that there is a clopen subset c of X such that $p \in c$ and, for $a \in A$ satisfying $a \leq e(c)$, $a \in I_r$ iff $a \in I_l$.

2.3. LEMMA. Suppose $p \in X$ and $\pi \in S_n$. Then π is compatible with p iff there is some $g \in G$ which is induced by π at p .

Proof. Suppose π induces g at p and $1 \leq r \leq n$. Let u be a neighbourhood of p such that $g(u_r)(q) = u_{\pi(r)}(q)$ for $q \in u$. Thus, for $q \in u$, $u_{\pi(r)}(q) = 0$ iff $g(u_r)(q) = 0$ iff $u_r(q) = 0$ since g induces an automorphism of B_q .

Conversely, suppose π is compatible with p . Choose a clopen neighbourhood c of p such that $u_r(q) = 0$ iff $u_{\pi(r)}(q) = 0$ for $1 \leq r \leq n$ and $q \in c$. Let $a = e(c)$. By 2.1 and the remark preceding this lemma, there is some $g \in G$ such that $g(u_r) = -a \cdot u_r + a \cdot u_{\pi(r)}$ for every r . This g is induced by π at p , since $a(p) = 1$ and hence $g(u_r)(p) = u_{\pi(r)}(p)$.

2.4. THEOREM. a) Let $X = \cup \{c_\pi \mid \pi \in S_n\}$ be a partition of X into pairwise disjoint clopen subsets such that, for every $p \in c_\pi$, π is compatible with p . Put $a_\pi = e(c_\pi)$ for $\pi \in S_n$. Then there is $g \in G$ such that, for $1 \leq r \leq n$,

$$g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}.$$

b) Conversely, let $g \in G$. Then there is a partition $X = \cup \{c_\pi \mid \pi \in S_n\}$ of X into pairwise disjoint clopen subsets such that, for $p \in c_\pi$, π is compatible with p , and $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$, where $a_\pi = e(c_\pi)$.

Proof. First note that $g \in G$, $a_\pi = e(c_\pi)$ where $(c_\pi \mid \pi \in S_n)$ is a partition of X and $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$ imply that π is compatible with p for $p \in c_\pi$: by $p \in c_\pi$, we get $a_\pi(p) = 1$ and $a_\rho(p) = 0$ for $\rho \in S_n$, $\rho \neq \pi$. So $g(u_r)(p) = u_{\pi(r)}(p)$, g is induced by π at p , and π is compatible with p .

To prove a), note that $\{a_\pi \cdot u_r \mid \pi \in S_n, 1 \leq r \leq n\}$ is a set of pairwise disjoint elements of B with supremum 1 and generating B over A . The existence of g follows by 2.1 and the remark preceding 2.3.

To prove b), let $g \in G$. For $\pi \in S_n$, put

$$v_\pi = \{p \in X \mid \pi \text{ induces } g \text{ at } p\}.$$

Each v_π is an open subset of X , and $X = \cup \{v_\pi \mid \pi \in S_n\}$: suppose $p \in X$. Define $\pi \in S_n$ as follows: let $1 \leq r \leq n$. If $u_r(p) = 0$, then $g(u_r)(p) = 0$; put $\pi(r) = r$. If $u_r(p) \neq 0$, $u_r(p)$ and hence $g(u_r)(p)$ is an atom of B_p ; let $\pi(r) = l$ where $g(u_r)(p) = u_l(p)$. Clearly, $p \in v_\pi$.

Since X is a Boolean space, there is a family $(c_\pi \mid \pi \in S_n)$ such that c_π is a clopen subset of v_π , $X = \cup \{c_\pi \mid \pi \in S_n\}$ and the c_π are pairwise disjoint. Put $a_\pi = e(c_\pi)$. Suppose $1 \leq r \leq n$ and $p \in X$, e.g. $p \in c_\pi$. Then $p \in v_\pi$ and

$$(\sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\})(p) = g(u_r)(p).$$

Theorem 2.4 says that the automorphisms of B over A are completely determined by certain partitions $(a_\pi \mid \pi \in S_n)$ of A resp. $(c_\pi \mid \pi \in S_n)$ of C . Unfortunately, for a given $g \in G$, a partition $(c_\pi \mid \pi \in S_n)$ defining g is not uniquely determined, since there may be different possibilities of choosing a clopen disjoint refinement of $(v_\pi \mid \pi \in S_n)$. We conclude this section by illustrating 2.4 by several examples.

If H is any group and A a BA, let X be the Stone space of A and

$$H[A] = \{f : X \rightarrow H \mid f \text{ is continuous}\}$$

where H is given the discrete topology. $H[A]$ is a subgroup of H^X and is usually called the bounded Boolean power of H by A . Recall that, for $B = A(u_1 \dots u_n)$, A and the subalgebra of B generated by u_1, \dots, u_n are independent iff $a \cdot u_r \neq 0$ for $a \in A \setminus \{0\}$, $1 \leq r \leq n$. A is then relatively complete in B . Conversely, suppose A is relatively complete in B . Then there is a partition $(a_k \mid 1 \leq k \leq n)$ of A (some of the a_k may equal zero) such that, for each k , the relative algebra $B \upharpoonright a_k = \{x \in B \mid x \leq a_k\}$ is generated over $A \upharpoonright a_k$ by k disjoint elements v_1, \dots, v_k which are independent from $A \upharpoonright a_k$: for $1 \leq r, l \leq n$, the set of those $p \in X$ such that $u_r(p) = u_l(p)$ is clopen. Hence, for $1 \leq k \leq n$, $c_k = \{p \in X \mid B_p \text{ has exactly } k \text{ atoms}\}$ is

clopen; put $a_k = e(c_k)$. By a compactness argument, construct $v_1, \dots, v_k \in B \upharpoonright a_k$ by patching together some of the u_r such that for $p \in c_k$, the atoms of B_p are $v_1(p), \dots, v_k(p)$.

2.5. EXAMPLE. Suppose $a \cdot u_r \neq 0$ for $1 \leq r \leq n$ and $a \in A \setminus \{0\}$. Then $\text{Aut}_A B \cong S_n[A]$.

Proof. Our assumption says that $u_r(p) \neq 0$ for each r and each $p \in X$. Hence each $\pi \in S_n$ is compatible with each $p \in X$ and, for fixed $g \in G$, the open sets v_π in the proof of 2.4 are disjoint, hence $c_\pi = v_\pi$. An isomorphism $\varphi : G \rightarrow S_n[A]$ is established by defining $\varphi(g)(p) = \pi$ iff $p \in v_\pi$.

2.6. EXAMPLE. Suppose A is relatively complete in B . Then there is a partition $(a_k \mid 1 \leq k \leq n)$ of A such that

$$\text{Aut}_A B \cong S_1[A \upharpoonright a_1] \times \dots \times S_n[A \upharpoonright a_n].$$

Proof. Choose, for $1 \leq k \leq n$, $a_k \in A$ as indicated above and let G_k be the Galois group of $B \upharpoonright a_k$ over $A \upharpoonright a_k$. Clearly,

$$\text{Aut}_A B \cong G_1 \times \dots \times G_n,$$

since $a_k \in A$. By 2.5, $G_k \cong S_k[A \upharpoonright a_k]$.

2.7. PROPOSITION. *The following conditions on (B, A) are equivalent:*

- a) A is relatively complete in B ;
- b) there is some $g \in G$ such that $g(b) \neq b$ for $b \in B \setminus A$;
- c) there is some finite subgroup H of G such that, for every $b \in B \setminus A$, there is some $g \in H$ satisfying $g(b) \neq b$.

Proof. Assume a). There is a finite partition T of C such that, for $1 \leq r \leq n$, $t \in T$ and $p, q \in t$, $u_r(p) = 0$ iff $u_r(q) = 0$. For $t \in T$, let $\pi_t \in S_n$ such that, for $p \in t$, $\pi_t(r) = r$ if $u_r(p) = 0$ and $u_r(p) \mapsto u_{\pi_t(r)}(p)$ is a cyclic permutation of the atoms of B_p which moves all these atoms. π_t is compatible with each $p \in t$; hence there is some $g \in G$ such that g is induced by π_t for $p \in t$, $t \in T$. Now let $b \in B \setminus A$. Choose $p \in X$, e.g. $p \in t$ where $t \in T$, such that $b(p) \notin \{0, 1\}$; put $b' = g(b)$. Let $At(B_p)$ be the set of atoms of B_p , $M = \{\alpha \in At(B_p) \mid \alpha \leq b(p)\}$, g_p the automorphism of B_p induced by g , $M' = \{g_p(\alpha) \mid \alpha \in M\}$. By the choice of π_t and g ,

$$b'(p) = g_p(b(p)) = \sum M' \neq \sum M = b(p)$$

which proves $b' \neq b$ — since, if π is a cyclic permutation of a finite set Y moving every element of Y and $M \subseteq Y$ satisfies $M = \{ \pi(m) \mid m \in M \}$, then $M = \emptyset$ or $M = Y$.

To prove that b) implies c) it is sufficient to know that every finitely generated subgroup of G is finite. We indicate a construction for finite subgroups of G . Let $T \subseteq C$ be a finite partition of C . A function $\varphi : T \rightarrow S_n$ is said to be compatible if, for every $t \in T$ and $p \in t$, $\varphi(t)$ is compatible with p . For each compatible $\varphi : T \rightarrow S_n$ let g_φ be the element of G mapping u_r to $\sum \{ e(t) \cdot u_{\varphi(t)(r)} \mid t \in T \}$. It is easily seen that

$$G_T = \{ g_\varphi \mid \varphi : T \rightarrow S_n \text{ compatible} \}$$

is a finite subgroup of G and that every finite subset of G is contained in some G_T .

Now suppose c), i.e. there is some finite subgroup H of G moving every $b \in B \setminus A$. We may assume that $H = G_T$ for some finite partition T of C . Assume that A is not relatively complete in B . By 2.2 there is some r such that I_r is not a principal ideal; w.l.o.g., $r = 1$. Let $\sigma = \{ p \in X \mid u_1(p) = 0 \}$. σ is a subset of X which is open but not closed; choose $p \in X$ which lies in the closure of σ but not in σ . W.l.o.g., for some k satisfying $1 \leq k \leq n$,

$$\{ r \mid 1 \leq r \leq n \text{ and } u_r \sim u_1 \text{ at } p \} = \{ 1, \dots, k \} .$$

Let c be a clopen neighbourhood of p such that, for $1 \leq r \leq k$ and $q \in c$, $u_r(q) = 0$ iff $u_1(q) = 0$. W.l.o.g., $c \in T$. There is some l such that $k < l \leq n$ and $u_l(p) \neq 0$; otherwise, let $c' \subseteq c$ a neighbourhood of p such that $u_l(q) = 0$ for $q \in c'$ and $k < l \leq n$. Choose $q \in c' \cap \sigma$ (since p lies in the closure of σ). In B_q , which has at least two elements, $1 = u_1(q) + \dots + u_n(q) = 0 + \dots + 0 = 0$, a contradiction. — Put $a = e(c)$ and $b = a \cdot u_1 + \dots + a \cdot u_k$. $b \in B \setminus A$, since $0 < b(p) = u_1(p) + \dots + u_k(p) < 1$ by our preceding claim. We prove that, for $g \in H = G_T$, $g(b) = b$, thus arriving at a final contradiction: there is some compatible $\varphi : T \rightarrow S_n$ such that $g = g_\varphi$. Consider $k \leq n$, $c \in T$ and $p \in c$ as constructed above. Since φ is compatible, $\pi = \varphi(c)$ is compatible with p ; hence π maps the set $\{ 1, \dots, k \}$ into itself, $g_\varphi(a \cdot u_r) = a \cdot u_{\pi(r)}$ for $1 \leq r \leq k$ (where $a = e(c)$) and $g(b) = b$.

3. TRUTH VALUES IN A FOR STATEMENTS ABOUT (B, A)

For the rest of this paper, let $\mathcal{L}_{BA} = \{+, \cdot, -, 0, 1\}$ the language of BA s and $\mathcal{L} = \mathcal{L}_{BA} \cup \{U\}$. Let T_{BAU} be the theory in \mathcal{L} such that the models of T_{BAU} have the form $(B, +, \cdot, -, 0, 1, A)$ where (B, \dots) is a BA and A is a subalgebra of B . We abbreviate a model (B, \dots, A) of T_{BAU} by $\mathcal{M} = (B, A)$. We assume the construction and notations of section 1. For each \mathcal{L} -formula $\varphi(x_1 \dots x_n)$ and $b_1, \dots, b_n \in B$, we defined

$$\|\varphi[b_1 \dots b_n]\| = \{p \in X \mid B_p \models \varphi[b_1(p) \dots b_n(p)]\}$$

where B_p abbreviates $(B_p, 2)$ and 2 is the two-element BA . Our first claim is that if $c = \|\varphi[b_1 \dots b_n]\|$ is a clopen subset of X for every φ , then $e(c) \in A$ is first-order definable in $\mathcal{M} = (B, A)$ from the parameters $b_1, \dots, b_n \in B$:

3.1. LEMMA. There is an effective procedure assigning to each formula $\varphi(x_1 \dots x_n)$ of \mathcal{L} a formula $s_\varphi(yx_1 \dots x_n)$ of \mathcal{L} (where y is a variable not occurring in φ) such that for $\mathcal{M} \models T_{BAU}$, properties (i) and (ii) are equivalent and (ii) implies (iii):

- (i) $\|\varphi[b_1 \dots b_n]\|$ is clopen for every $\varphi(x_1 \dots x_n)$ in \mathcal{L} and $b_1, \dots, b_n \in B$;
- (ii) $\mathcal{M} \models \forall x_1 \dots \forall x_n \exists y s_\varphi(yx_1 \dots x_n)$ for every $\varphi(x_1 \dots x_n)$ in \mathcal{L} ;
- (iii) if $b_1, \dots, b_n \in B$, then $a = e(c)$ where $c = \|\varphi[b_1 \dots b_n]\|$ is the unique element b of B such that $\mathcal{M} \models s_\varphi[bb_1 \dots b_n]$.

Proof. The inductive definition of s_φ will show that (i) is equivalent to (ii) and (i) implies (iii), the interesting cases being φ atomic or φ existential. In both cases the fact that $\|\varphi[\dots]\|$ is clopen will be expressed by stating " $a (= e(\|\varphi[\dots]\|))$ is the largest element of A such that $e^{-1}(a) \subseteq \|\varphi[\dots]\|$ ". This includes, if φ has the form $\exists x\psi$, the maximum principle for the Boolean valuation

$$\psi, b_1 \dots b_n \rightarrow \|\psi[b_1 \dots b_n]\|$$

of \mathcal{M} in C : there is some $b \in B$ such that

$$\|\psi[b'b_1 \dots b_n]\| \leq \|\psi[bb_1 \dots b_n]\|$$

for every $b' \in B$, and hence $\|\psi[bb_1 \dots b_n]\| = \|\exists x\psi[xb_1 \dots b_n]\|$. We now proceed to define the formulas s_φ .

a) Suppose φ is an atomic formula of \mathcal{L}_{BA} , i.e. φ has the form $t_1(x_1 \dots x_n) = t_2(x_1 \dots x_n)$ where t_1, t_2 are terms in \mathcal{L}_{BA} . Let $s_\varphi(yx_1 \dots x_n)$ be the formula

$$U(y) \wedge y \cdot t_1 = y \cdot t_2 \wedge \forall y' (U(y') \wedge y' \cdot t_1 = y' \cdot t_2 \rightarrow y' \leq y).$$

b) Suppose φ has the form $U(t(x_1 \dots x_n))$ where t is a term in \mathcal{L}_{BA} . Let ψ, χ be the atomic \mathcal{L}_{BA} -formulas " $t = 1$ " resp. " $t = 0$ ". Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)].$$

c) Suppose φ has the form $\neg \psi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 [y = -y_1 \wedge s_\psi(y_1 x_1 \dots x_n)].$$

d) Suppose φ has the form $\psi(x_1 \dots x_n) \vee \chi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)].$$

e) Suppose φ has the form $\exists x \psi(x x_1 \dots x_n)$. Let s_φ be the formula

$$\exists x s_\psi(y x x_1 \dots x_n) \wedge \forall x' \forall y' [s_\psi(y' x' x_1 \dots x_n) \rightarrow y' \leq y].$$

Let σ be the \mathcal{L}_{BA} -formula stating that the supremum of the atoms of a BA exists; σ^U is the relativization of σ to the one-place predicate U of \mathcal{L} . The models of $T_{BA} \cup \{\sigma\}$ are called separated BA s in [3]. Let T be the \mathcal{L} -theory

$$T = T_{BAU} \cup \left\{ \forall x_1 \dots \forall x_n \exists y s_\varphi(y x_1 \dots x_n) \mid \varphi(x_1 \dots x_n) \text{ in } \mathcal{L} \right\} \\ \cup \left\{ \sigma^U, s_\sigma(1) \right\}.$$

The last two axioms of T express, for a model $\mathcal{M} = (B, A)$ of T_{BAU} , that A and each stalk B_p are separated BA s. Let \mathbf{K} be the class of \mathcal{L} -structures $\mathcal{M} = (B, A)$ where B is a cBA and A is relatively complete in B . We shall prove in section 4 that T is an axiomatization of the first-order theory of \mathbf{K} . The easy part of this is:

3.2. THEOREM. *Each structure \mathcal{M} in \mathbf{K} is a model of T .*

Proof. Let $\mathcal{M} = (B, A) \in \mathbf{K}$, i.e. B is complete and A is relatively complete in B . Hence $\mathcal{M} \models T_{BAU}$ and A is a separated BA . By 1.1, $\|\varphi[b_1 \dots b_n]\|$ is clopen for every atomic formula φ of \mathcal{L} and arbitrary $b_1, \dots, b_n \in B$. If $\|\varphi[b_1 \dots b_n]\|$ and $\|\psi[b_1 \dots b_n]\|$ are clopen subsets of X , so are $\|\neg \varphi[b_1 \dots b_n]\|$ and $\|(\varphi \vee \psi)[b_1 \dots b_n]\|$. Hence we assume that φ

has the form $\exists x \psi (xx_1 \dots x_n)$ and that $\| \psi [bb_1 \dots b_n] \|$ is clopen for fixed $b_1, \dots, b_n \in B$ and arbitrary $b \in B$. For the rest of the proof, we omit the parameters b_1, \dots, b_n . Let

$$u = \cup \{ \| \psi [\beta] \| \mid \beta \in B \}.$$

By our inductive assumption, u is an open subset of X . Choose, by Zorn's lemma, a maximal family $F = \{ (b_i, c_i) \mid i \in I \}$ such that $b_i \in B$, c_i is a clopen subset of u , $c_i \subseteq \| \psi [b_i] \|$, $i \neq j$ implies $c_i \cap c_j = \phi$. It follows that c , the closure of $\cup_{i \in I} c_i$, includes u (by maximality of F). A is a cBA ,

hence X is extremally disconnected and c is clopen. By completeness of B , there is some $b \in B$ such that $b \cdot e(c_i) = b_i$ for $i \in I$. Thus, for $i \in I$, $c_i \subseteq \| \psi [b] \|$. So, for $\beta \in B$, $\| \psi [\beta] \| \subseteq u \subseteq c \subseteq \| \psi [b] \| = \| \exists x \psi (x) \|$.

Finally we show that B_p is separated for each $p \in X$. Let $\alpha(x)$ be the \mathcal{L}_{BA} -formula stating that x is an atom and let $\beta(x)$, $\gamma(x)$ be the \mathcal{L}_{BA} -formulas $\alpha(x) \vee x = 0$ resp. $\forall y (\alpha(y) \rightarrow y \leq x)$. Put $M = \{ f \in B \mid \| \beta [f] \| = 1 \|$ and let b be the supremum of M in B . We show that $b(p)$ is, for each $p \in X$, the supremum of the atoms of B_p .

First suppose $s \in B_p$ is an atom of B_p . There is some $f \in M$ such that $f(p) = s$ (note that $\| \alpha [f] \|$ is clopen for each $f \in B$). So $f \leq b$ and $s = f(p) \leq b(p)$. — On the other hand, suppose $t \in B_p$ and $s \leq t$ for every atom s of B_p . Choose $g \in B$ such that $g(p) = t$. Then $p \in c = \| \gamma [g] \|$. For $f \in M$, $e(c) \cdot f \leq g$, since $q \in c$ implies that $f(q)$ is zero or an atom of B_q and thus $f(q) \leq g(q)$. By the definition of b , $e(c) \cdot b \leq g$. This implies (by $p \in c$) $b(p) \leq g(p) = t$.

4. DECIDABILITY AND COMPLETIONS OF $Th(\mathbf{K})$

Call $T_{sBA} = T_{BA} \cup \{ \sigma \}$ the theory of separated BA s, where T_{BA} is the theory of BA s and σ was defined in section 3. We give a short review of the completions of T_{sBA} . Let, for $n \in \omega$, φ_n be the \mathcal{L}_{BA} -sentence stating that there are exactly n atoms and ψ the \mathcal{L}_{BA} -sentence stating that there is a non-zero atomless element. Let $\chi_n = \neg (\varphi_0 \vee \dots \vee \varphi_{n-1})$; so χ_n says that there are at least n atoms. Define, for $n \in \omega + 1$ and $i \in 2 = \{ 0, 1 \}$, an \mathcal{L}_{BA} -theory T_{ni} by

$$\begin{aligned} T_{n0} &= T_{sBA} \cup \{ \varphi_n, \neg \psi \} \\ T_{n1} &= T_{sBA} \cup \{ \varphi_n, \psi \} \end{aligned}$$

for $n \in \omega$, and

$$T_{\omega 0} = T_{sBA} \cup \{ \chi_n \mid n \in \omega \} \cup \{ \neg \psi \}$$

$$T_{\omega 1} = T_{sBA} \cup \{ \chi_n \mid n \in \omega \} \cup \{ \psi \}.$$

Put $\tau = \{ T_{ni} \mid n \in \omega + 1, i \in 2 \}$. It is then clear that each separated BA satisfies exactly one of the theories in τ , and for each $t \in \tau$ there is a cBA satisfying t . Moreover, any two models of any $t \in \tau$ are elementarily equivalent by 5.5.10 in [1]. Thus the theories $t \in \tau$ are just the completions of T_{sBA} and can be thought of as being the elementary equivalence types of separated BAs or $cBAs$. Moreover, an \mathcal{L}_{BA} -sentence holds in every separated BA iff it holds in every cBA . The following proposition is essential for the main theorems of this section:

4.1. PROPOSITION. *Let $s, t \in \tau$. Then there is a structure (B, A) in \mathbf{K} such that A is a model of s and each stalk B_p is a model of t .*

Proof. By the above remarks, choose $cBAs$ A and F which are models of s resp. t . Let $A * F$ be the free product of A and F . Thus A is relatively complete in $A * F$ and each stalk $(A * F)_p$, where p is an ultrafilter of A , is easily seen to be isomorphic to F , hence a model of t . Unfortunately, $A * F$ is incomplete unless A or F is finite. So let $B = (A * F)^*$ be the completion of $A * F$; note that $A * F$ is a dense subalgebra of B . $(B, A) \in \mathbf{K}$, since the inclusion maps from A to $A * F$ and from $A * F$ to B are complete. For $p \in X$ (the Stone space of A), B_p is a separated BA by 3.2 but in general a proper extension of $(A * F)_p$. We show, with the notations of section 1, that B_p is elementarily equivalent to F . For the following proof of this, recall that, for $f \in F \setminus \{0\}$ and $p \in X$, $\pi_p(f) = f(p) \neq 0$ since F is independent from A in $A * F \subseteq B$. Thus, the restriction of $\pi_p : B \rightarrow B_p$ to F is a monomorphism. The elementary equivalence of B_p and F is established by the following four claims.

Claim 1. For each atom f of F , $f(p)$ is an atom of B_p (hence, if F has at least n atoms, where $n \in \omega$, then B_p has at least n atoms): clearly, $f(p) > 0$ for $p \in X$. Assume that

$$u = \{ p \in X \mid f(p) \text{ is not an atom of } B_p \}$$

is non-empty. By 3.2, u is a clopen subset of X . Choose, by the maximum principle stated in section 3, $b \in B$ such that $b(p) = 0$ for $p \notin u$ and $0 < b(p) < f(p)$ for $p \in u$. Since $b > 0$, choose $a \in A$ and $g \in F$ such that $0 < a \cdot g \leq b$; let $p \in X$ such that $a(p) \cdot g(p) \neq 0$. Thus $p \in u$, $a(p) = 1$, and

$0 < g(p) \leq b(p) < f(p)$. It follows that $0 < g < f$, contradicting the fact that f was an atom of F .

Claim 2. If B_p has at least n atoms, where $1 \leq n < \omega$, then F has at least n atoms: assume that M is a subset of $At(B_p)$, the set of atoms of B_p , such that M has exactly n elements but $At(F)$ has at most $n - 1$ elements. We prove:

(a) Let $x \in M$. Then there is $f_x \in At(F)$ such that $f_x(p) = x$.

Claim 2 follows from (a), since the assignment of f_x to x is injective. And (a) will follow from

(b) Let $x \in M$, u a clopen neighbourhood of p such that, w.l.o.g., for $q \in u$, B_q has at least one atom. Let $b \in B$ such that, for $q \notin u$, $b(q) = 0$ and for $q \in u$, $b(q)$ is an atom of B_q , and $b(p) = x$. Then there are $q \in u$ and $f \in At(F)$ such that $f(q) = b(q)$. (Hence $At(F)$ is non-empty).

Proof of (b). By $b > 0$, choose $a \in A$, $f \in F$ such that $0 < a \cdot f \leq b$. Since $b(q) = 0$ for $q \notin u$, there is some $q \in u$ such that $a(q) \cdot f(q) \neq 0$, which implies $0 < f(q) \leq b(q)$. $f(q) = b(q)$, since $b(q)$ is an atom of B_q . Finally $f \in At(F)$, since a splitting of f in F into two non-zero disjoint elements would give rise to a splitting of $b(q)$ in B_q .

Proof of (a). Let $x \in M$ and choose u and b as in (b). Assume (a) is false. Then, for each $f \in At(F)$, $f(p) \neq x = b(p)$; by finiteness of $At(F)$, there is a clopen neighbourhood v of p such that, for $q \in v$ and $f \in At(F)$, $b(q) \neq f(q)$. Let $c \in B$ such that $c(q) = 0$ for $q \notin v$ and $c(q) = b(q)$ for $q \in v$. This contradicts (b), applied to v and c instead of u and b .

Claim 3. If F has a non-zero atomless element f (which means that $F \upharpoonright f$ is atomless), then each B_p has a non-zero atomless element x : let $x = \pi_p(f)$. $x > 0$, since π_p is one-one on F . $F \upharpoonright f$ and hence, by Claim 2, $(B \upharpoonright f)_p$ is atomless. So $B_p \upharpoonright x = \pi_p(B \upharpoonright f) = (B \upharpoonright f)_p$ is atomless.

Claim 4. If B_p has a non-zero atomless element x , then F has a non-zero atomless element f : assume that F is atomic. Let

$$u = \{q \in X \mid B_q \text{ is not atomic}\}.$$

u is a clopen neighbourhood of p . By the maximum principle, choose $b \in B$ such that $b(q) = 0$ for $q \notin u$, $b(q)$ is a non-zero atomless element of

B_q for $q \in u$, $b(p) = x$. Choose $a \in A$, $g \in F$ such that $0 < a \cdot g \leq b$; w.l.o.g., g is an atom of F . Choose $q \in X$ such that $a(q) \cdot g(q) \neq 0$. Thus $q \in u$ and $g(q) \leq b(q)$. By Claim 1, $g(q)$ is an atom of B_q , contradicting the choice of $b(q)$.

4.2. REMARK. Suppose that, for every i in an index set I , $\mathcal{M}_i = (B_i, A_i)$ is an element of \mathbf{K} . Then $\mathcal{M} = (B, A)$, where $B = \prod_{i \in I} B_i$ and $A = \prod_{i \in I} A_i$, is in \mathbf{K} . Let $\varphi(x_1 \dots x_k)$ be an \mathcal{L} -formula and $b_1, \dots, b_k \in B$, $b_j = (b_{ij})_{i \in I}$. Put $a_i = e(\|\varphi[b_{i1} \dots b_{ik}]\|^{M_i})$. Then

$$e(\|\varphi[b_1 \dots b_k]\|^{M}) = (a_i)_{i \in I}.$$

Proof. By induction on the complexity of φ .

We shall need the following Feferman-Vaught theorem about sheaves over Boolean spaces from [2]:

4.3. THEOREM (Comer). *Let \mathcal{L} be an arbitrary language. There is an effective assignment*

$$\varphi(x_1 \dots x_k) \mapsto (\Phi; \mathfrak{D}_1, \dots, \mathfrak{D}_m)$$

for \mathcal{L} -formulas $\varphi(x_1 \dots x_k)$ such that

- a) $\mathfrak{D}_1, \dots, \mathfrak{D}_m$ are \mathcal{L} -formulas having at most the free variables $x_1 \dots x_k$, and

$$\models (\bigvee_{1 \leq i \leq m} \mathfrak{D}_i) \wedge \bigwedge_{1 \leq i < j \leq m} \neg(\mathfrak{D}_i \wedge \mathfrak{D}_j)$$

- b) Φ is an \mathcal{L}_{BA} -formula having at most the free variables $y_1 \dots y_m$;
- c) for each sheaf $\mathcal{S} = (S, \pi, X, \mu)$ of \mathcal{L} -structures such that X is a Boolean space and $\|\psi[f_1 \dots f_n]\|$ is a clopen subset of X for every $\psi(x_1 \dots x_n)$ in \mathcal{L} and $f_1, \dots, f_n \in \Gamma(\mathcal{S})$: if $b_1, \dots, b_k \in \Gamma(\mathcal{S})$, then

$$\Gamma(\mathcal{S}) \models \varphi[b_1 \dots b_k] \text{ iff } C \models \Phi[c_1 \dots c_m],$$

where C is the BA of clopen subsets of X and $c_i = \|\mathfrak{D}_i[b_1 \dots b_k]\|$.

For two separated BAs A and A' , let I be the set of partial functions f from A to A' such that $\text{dom}(f) = \{a_1, \dots, a_n\}$ is a finite partition of A (where some of the a_i may be zero), $\text{rge}(f) = \{a'_1, \dots, a'_n\}$ where $a'_i = f(a_i)$ is a partition of A' , and every $A \upharpoonright a_i$ is elementarily equivalent

to $A' \vdash a_i'$. If A, A' are \aleph_1 -saturated or σ -complete, the following conditions are equivalent:

- a) $A \equiv A'$;
- b) I is non-empty;
- c) I has the back-and-forth property.

Moreover, if $f \in I$ is as above and A, A' are arbitrary separated BA s, then $(A, a_1, \dots, a_n) \equiv (A', a_1', \dots, a_n')$.

Let T_{sBA2} be the \mathcal{L} -theory

$$T_{sBA2} = T_{sBA} \cup \{ \forall x (U(x) \leftrightarrow x = 0 \vee x = 1) \}.$$

Since T_{BA} is decidable, T_{sBA} and T_{sBA2} are decidable.

4.4. THEOREM. *There is an effective procedure deciding for every \mathcal{L} -sentence φ whether $T \vdash \varphi$. Moreover, $T \vdash \varphi$ if and only if φ holds in every model \mathcal{M} in \mathbf{K} .*

Proof. Let φ be given. Construct $(\Phi(y_1 \dots y_m); \mathfrak{g}_1, \dots, \mathfrak{g}_m)$ by 4.3. For every i such that $1 \leq i \leq m$, decide whether $T_{sBA2} \vdash \neg \mathfrak{g}_i$. W.l.o.g., assume that $T_{sBA2} \cup \{ \mathfrak{g}_i \}$ is consistent for $1 \leq i \leq r$ and inconsistent for $r + 1 \leq i \leq m$. By $\vdash \mathfrak{g}_1 \vee \dots \vee \mathfrak{g}_m$, we have $1 \leq r$ (it is possible that $r = m$). Next, construct the formula

$$\Phi'(y_1 \dots y_m) = \left(\bigwedge_{r+1 \leq i \leq m} (y_i = 0) \rightarrow \Phi(y_1 \dots y_m) \right).$$

We show the equivalence of

- a) $T \vdash \varphi$;
- b) $\mathcal{M} \models \varphi$ for every $\mathcal{M} \in \mathbf{K}$;
- c) $T_{sBA} \vdash \forall y_1 \dots \forall y_m \Phi'(y_1 \dots y_m)$.

Then, by decidability of T_{sBA} , T is decidable and 4.4 is proved. *a)* implies *b)* by 3.2. To prove that *c)* implies *a)*, assume there is $\mathcal{M} \models T$ such that $\mathcal{M} \not\models \varphi$, e.g. $\mathcal{M} = (B, A)$. Put $a_i = e(\|\mathfrak{g}_i\|^{\mathcal{M}})$. By 4.3 and $\mathcal{M} \not\models \varphi$, we see $A \not\models \Phi[a_1 \dots a_m]$. By our choice of $r \leq m$, we get $a_{r+1} = \dots = a_m = 0$. Thus $A \not\models \Phi'[a_1 \dots a_m]$ and *c)* is false. Now assume *c)* does not hold; we show that *b)* is false. Let A' be a separated BA and $a_1', \dots, a_m' \in A'$ such that $a_{r+1}' = \dots = a_m' = 0$ and $A' \not\models \Phi[a_1' \dots a_m']$. W.l.o.g., $a_i' \neq 0$ for $1 \leq i \leq r$. By choice of r , there are $t_1, \dots, t_r \in \tau$ such that $t_i \models \mathfrak{g}_i$ for $1 \leq i \leq r$.

Let, for these i, s_i be the element of τ such that $A' \upharpoonright a_i' \models s_i$. By 4.1, there are $\mathcal{M} = (B, A) \in \mathbf{K}$ and $a_1, \dots, a_r \in A$ such that $1 = a_1 + \dots + a_r$, $a_i \cdot a_j = 0$ for $1 \leq i < j \leq r$, $A \upharpoonright a_i \models s_i$ and $(B \upharpoonright a_i)_p \models t_i$ for those $p \in X$ satisfying $a_i(p) = 1$. So $e(\|\mathfrak{g}_i\|^\mathcal{M}) = a_i$ by 4.2. Put $a_{r+1} = \dots = a_m = 0$. It follows that $(A, a_1, \dots, a_m) \equiv (A', a_1', \dots, a_m')$, $A \not\models \Phi[a_1 \dots a_m]$ and $\mathcal{M} \not\models \varphi$ by 4.3.

In the next theorem, we characterize elementary equivalence of models of T . Call the following sentences in \mathcal{L}_{BA} basic sentences: $\varphi_n \wedge \psi$, $\varphi_n \wedge \neg \psi$, $\chi_n \wedge \psi$, $\chi_n \wedge \neg \psi$ (where $n \in \omega$). It follows by the analysis of the completions of T_{sBA} given in the beginning of this section that for each \mathcal{L}_{BA} -sentence \mathfrak{g} there are basic sentences β_1, \dots, β_n such that

$$T_{sBA} \vdash (\mathfrak{g} \leftrightarrow \bigvee_{i=1}^n \beta_i) \wedge \bigwedge_{1 \leq i < j \leq n} \neg (\beta_i \wedge \beta_j).$$

This fact is easily extended to T_{sBA2} : by replacing each atomic formula $U(t)$ where t is a term in \mathcal{L}_{BA} by " $t = 0 \vee t = 1$ ", we see that for each \mathcal{L} -sentence \mathfrak{g} there are basic sentences β_1, \dots, β_n satisfying

$$T_{sBA2} \vdash (\mathfrak{g} \leftrightarrow \bigvee_{i=1}^n \beta_i) \wedge \bigwedge_{1 \leq i < j \leq n} \neg (\beta_i \wedge \beta_j).$$

Now, if β, γ are basic sentences, let $\sigma_{\beta\gamma}$ be the following \mathcal{L} -sentence:

$$\sigma_{\beta\gamma} = \exists y (\gamma^y \wedge s_\beta(y)),$$

where $s_\beta(y)$ is the \mathcal{L} -formula assigned to β in 3.1 and γ^y is the result of relativizing the quantifiers $\exists x\varphi \dots$ in γ to $\exists x(U(x) \wedge x \leq y \wedge \varphi^y \dots)$. A model (B, A) of T satisfies $\sigma_{\beta\gamma}$ iff $A \upharpoonright a \models \gamma$, where $a = e(c)$ and $c = \|\beta\|$.

4.5. THEOREM. Let $\mathcal{M} = (B, A)$, $\mathcal{M}' = (B', A')$ be models of T . Then \mathcal{M} is elementarily equivalent to \mathcal{M}' if and only if, for any basic sentences β, γ ,

$$\mathcal{M} \models \sigma_{\beta\gamma} \text{ iff } \mathcal{M}' \models \sigma_{\beta\gamma}.$$

Proof. The only-if-part is clear. Suppose that \mathcal{M} and \mathcal{M}' satisfy the same sentences of the form $\sigma_{\beta\gamma}$. Let φ be an \mathcal{L} -sentence and $\mathcal{M} \models \varphi$; we want to show that $\mathcal{M}' \models \varphi$. Let $(\Phi(y_1 \dots y_m); \mathfrak{g}_1, \dots, \mathfrak{g}_m)$ be the sequence assigned to φ by 4.3; every \mathfrak{g}_i is an \mathcal{L} -sentence. Put $a_i = e(\|\mathfrak{g}_i\|^\mathcal{M})$; by 4.3 and $e : C \rightarrow A$ being an isomorphism, we have that $\{a_1, \dots, a_m\}$

is a partition of A and $A \models \Phi [a_1 \dots a_m]$. In the same way, put $a'_i = e' (\| \mathcal{G}_i \| \mathcal{M}')$; $\{a'_1, \dots, a'_m\}$ is a partition of A' . It is sufficient to show that $(A, a_1, \dots, a_m) \equiv (A', a'_1, \dots, a'_m)$, for this implies $A' \models \Phi [a'_1 \dots a'_m]$ and finally $\mathcal{M}' \models \varphi$ by 4.3.

For every \mathcal{G}_i , choose basic sentences $\beta_{i1}, \dots, \beta_{in_i}$ such that

$$T_{sBA2} \vdash (\mathcal{G}_i \leftrightarrow \bigvee_j \beta_{ij} \wedge \bigwedge_{j < l} \neg (\beta_{ij} \wedge \beta_{il}).$$

Put $\alpha_{ij} = e (\| \beta_{ij} \| \mathcal{M})$, $\alpha'_{ij} = e' (\| \beta_{ij} \| \mathcal{M}')$ for $1 \leq i \leq m$, $1 \leq j \leq n_i$. Then a_i is the disjoint sum of the α_{ij} ($1 \leq j \leq n_i$), a'_i is the disjoint sum of the α'_{ij} ($1 \leq j \leq n_i$). For every i, j ,

$$A \upharpoonright \alpha_{ij} \equiv A' \upharpoonright \alpha'_{ij} :$$

let γ be any basic sentence of \mathcal{L}_{BA} and assume $A \upharpoonright \alpha_{ij} \models \gamma$; we want to show that $A' \upharpoonright \alpha'_{ij} \models \gamma$. But $A \upharpoonright \alpha_{ij} \models \gamma$ means that $\mathcal{M} \models \sigma_{\beta_{ij}\gamma}$. By our main assumption, $\mathcal{M}' \models \sigma_{\beta_{ij}\gamma}$ and $A' \upharpoonright \alpha'_{ij} \models \gamma$.

We have shown that the partial function f mapping α_{ij} to α'_{ij} is an element of the set of back-and-forth-isomorphisms defined after 4.3. Hence,

$$(A, \alpha_{11}, \dots, \alpha_{mn_m}) \equiv (A', \alpha'_{11}, \dots, \alpha'_{mn_m})$$

and

$$(A, a_1, \dots, a_m) \equiv (A', a'_1, \dots, a'_m).$$

We shall finally describe the completions of T by giving a one-one correspondance between a set P (consisting of pairs of mappings from $\omega \times 2$ to $(\omega+1) \times 2$) and these completions. For $m, m' \in \omega+1$ and $j, j' \in 2$, define

$$(m, j) + (m', j') = (m'', j'')$$

where m'' is the cardinal sum of m and m' and j'' is the maximum of j and j' . Let

$$P = \{ (\alpha, \rho) \mid \alpha, \rho : \omega \times 2 \rightarrow (\omega+1) \times 2 \text{ and, for } (n, i) \in \omega \times 2, \rho(n, i) = \rho(n+1, i) + \alpha(n, i) \}.$$

In the following definition, we refer to the \mathcal{L}_{BA} -theories T_{ni} defined in the beginning of this section. For $(\alpha, \rho) \in P$, let $T_{\alpha\rho}$ the \mathcal{L} -theory

$$\begin{aligned} T_{\alpha\rho} = & T \cup \{ \exists x (\sigma_{(\varphi_n \wedge \neg \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\alpha(n,0)} \} \\ & \cup \{ \exists x (\sigma_{(\chi_n \wedge \neg \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\rho(n,0)} \} \\ & \cup \{ \exists x (\sigma_{(\varphi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\alpha(n,1)} \} \\ & \cup \{ \exists x (\sigma_{(\chi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\rho(n,1)} \}. \end{aligned}$$

If $\mathcal{M} = (B, A)$ is a model of T , then $\mathcal{M} \models T_{\alpha\rho}$ iff, for $a_1 = e(\|\varphi_n \wedge \neg \psi\|^{\mathcal{M}})$ $A \upharpoonright a_1 \models T_{\alpha(n,0)}$, ..., for $a_4 = e(\|\chi_n \wedge \psi\|^{\mathcal{M}})$, $A \upharpoonright a_4 \models T_{\rho(n,1)}$.

4.6. THEOREM. $\{T_{\alpha\rho} \mid (\alpha, \rho) \in P\}$ is the set of completions of T . Moreover, each $T_{\alpha\rho}$ has a model in \mathbf{K} .

Proof. If (α, ρ) and (α', ρ') are different elements of P , then $T_{\alpha\rho} \cup T_{\alpha'\rho'}$ is inconsistent (recall that every T_{mj} , where $m \in \omega + 1, j \in 2$, is complete in \mathcal{L}_{BA}). If \mathcal{M} is a model of T , there is some $(\alpha, \rho) \in P$ such that $\mathcal{M} \models T_{\alpha\rho}$ (e.g., put $a_1 = e(\|\varphi_n \wedge \neg \psi\|^{\mathcal{M}})$ and let $\alpha(n, 0)$ be the pair $(k, j) \in (\omega + 1) \times 2$ such that $A \upharpoonright a_1 \models T_{kj}$, etc.). If $(\alpha, \rho) \in P$ and $\mathcal{M}, \mathcal{M}'$ are models of $T_{\alpha\rho}$, then \mathcal{M} and \mathcal{M}' are elementarily equivalent by 4.5, since $T_{\alpha\rho}$ says which sentences of the form $\sigma_{\beta\gamma}$ are satisfied in \mathcal{M} and \mathcal{M}' . So it is sufficient to prove that each $T_{\alpha\rho}$ has a model which lies even in \mathbf{K} .

For simplicity, we construct $\mathcal{M} \in \mathbf{K}$ satisfying the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$ — for, if $\mathcal{N} \in \mathbf{K}$ satisfies the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,1)}$ and $T_{\rho(n,1)}$, then $\mathcal{M} \times \mathcal{N} \in \mathbf{K}$ is a model of $T_{\alpha\rho}$. Abbreviate $\alpha(n, 0)$ by t_n , $\rho(n, 0)$ by s_n . We first construct a complete BA A and a sequence $(a_n)_{n \in \omega}$ in A such that the a_n are pairwise disjoint and

$$(*) \quad A \upharpoonright a_n \models t_n, \quad A \upharpoonright r_n \models s_n$$

where $r_n = -(a_0 + \dots + a_{n-1})$. Let A be a complete BA which is a model of s_0 . Suppose $a_0, \dots, a_{n-1} \in A$ are pairwise disjoint and a_0, \dots, a_{n-1}, r_n satisfy (*). Since $s_n = s_{n+1} + t_n$, $A \upharpoonright r_n \models s_n$ and A is complete, there are a_n and $r_{n+1} \in A$ such that $r_n = a_n + r_{n+1}$, $a_n \cdot r_{n+1} = 0$, $A \upharpoonright a_n \models t_n$ and $A \upharpoonright r_{n+1} \models s_{n+1}$. — Finally, let $a_\omega = -\sum_{n \in \omega} a_n$. By the proof of 4.1,

there is, for $n \in \omega$, $\mathcal{M}_n = (B_n, A_n) \in \mathbf{K}$ such that $A_n = A \upharpoonright a_n$ and each stalk $(B_n)_p$ of the sheaf representation of \mathcal{M}_n is a model of $\varphi_n \wedge \neg \psi$. Moreover there is $\mathcal{M}_\omega = (B_\omega, A_\omega) \in \mathbf{K}$ such that $A_\omega = A \upharpoonright a_\omega$ and each stalk $(B_\omega)_p$ of the sheaf representation of \mathcal{M}_ω is a model of $T_{\omega 0}$. Put $\mathcal{M} = (B, A)$ where B is a complete BA which lies over A as $\prod_{n \in \omega} B_n$ lies over $\prod_{n \in \omega} A_n$. By 4.2, $e(\|\varphi_n \wedge \neg \psi\|^{\mathcal{M}}) = a_n$ and $e(\|\chi_n \wedge \neg \psi\|^{\mathcal{M}}) = r_n$; so \mathcal{M} is a model of the part of $T_{\alpha\rho}$ referring to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$.

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