Zeitschrift: L'Enseignement Mathématique

Band: 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FROBENIUS RECIPROCITY AND LIE GROUP REPRESENTATIONS ON $\bar{\delta}$ COHOMOLOGY SPACES

Autor: Williams, Floyd L.

DOI: https://doi.org/10.5169/seals-52231

FROBENIUS RECIPROCITY AND LIE GROUP REPRESENTATIONS ON $\bar{\partial}$ COHOMOLOGY SPACES [1])

by Floyd L. WILLIAMS

To the memory of Dr. Walter R. Talbot

1. INTRODUCTION

The general theme of this lecture is Lie group representations on complex $\bar{\partial}$ cohomology spaces and extensions thereby of the classical Frobenius reciprocity theorem. Particular but not exclusive attention will be focused on the representations of semisimple groups. We shall survey some results ranging, historically, from the Borel-Weil-Bott-Kostant theorem to the recent theorem of W. Schmid which confirms the Kostant-Langlands conjecture. We shall also discuss, along these lines, recent results of Moscovici, Verona, Rosenberg and Penney for nilpotent groups. Before developing the particular ideas of the lecture we begin with some broader remarks which may serve as a more general frame of reference.

The finite dimensional representation theory of compact semisimple Lie groups is now a well established chapter in classical mathematics. The theory is due to E. Cartan and H. Weyl [16], [93]. Using non-algebraic methods, Weyl showed the complete reducibility of all (finite dimensional) representations. That is, every representation is the direct sum of irreducible representations. A modern algebraic proof of this fact can be accomplished using Lie algebra cohomology. Cartan classified the irreducible representations by setting up a 1-1 correspondence with the equivalence classes of such representations and the so-called dominant integral linear forms on a Cartan subalgebra of the Lie algebra of the group. This is the celebrated "highest weight" theory. Cartan's case by case approach depended on the classification of the simple Lie algebras. A more

[1]) This is an expanded version of an invited address delivered at the 769th meeting of the American Mathematical Society in Washington D.C. on October 20, 1979.

uniform and elegant treatment was given later by Harish-Chandra [18] and C. Chevalley (unpublished), independently. Harish-Chandra's arguments were simplified by N. Jacobson [48].

Another triumph of the theory was the derivation by Weyl of his famous character formula [93]. This gives in particular the dimension of an irreducible module in terms of its highest weight and it permits an explicit determination of the Plancherel theorem, and thus the harmonic analysis, for compact Lie groups. As first noted by Bott and Kostant [12], [50], the character formula may also be derived by cohomological considerations. It is in fact, in the proper context, an Euler-Poincaré formula expressing the equality of an alternating sum of traces on the cochain and cohomology level.

Given a compact semisimple group K and the irreducible representations of it, corresponding to highest weights, there naturally arises the question of realizing these representations in some concrete manner. By the principle of analytic continuation (Weyl's "unitary trick") we may consider equivalently the irreducible holomorphic representations of the complexification $K^{\mathbf{C}}$, when K is simply connected. One particular realization is given by the classical Borel-Weil theorem [11], [12] which asserts, in effect, that all such representations of $K^{\mathbf{C}}$ occur as the modules of holomorphic sections of appropriate homogeneous line bundles. That is, the modules occur "in zero dimensional cohomology". The Borel-Weil theorem may be viewed therefore as a type of imprimitivity theorem in the frame-work of holomorphic induction. We remark that the irreducible representations of compact groups definitely are *not* induced in the sense of Frobenius and Mackey, although the corresponding statement is valid, say for simply connected nilpotent Lie groups. However *in a holomorphic context* analogues of the Frobenius identity remain valid. [1]) The validity of this identity for non-compact groups as well for representations "in higher cohomology" is precisely the focal point of this lecture.

For a non-compact semisimple group G one must consider infinite dimensional representations (in contrast with the compact case there are no finite dimensional unitary representations of such a group other than the trivial one-dimensional representation) for the purposes of harmonic analysis. Much attention has been devoted in this direction in recent years. The foundations here have been layed principally by Harish-Chandra in some profoundly deep work. In a long and deep study of the local structure of invariant eigendistributions, which he showed to be locally integrable functions which are even analytic on a dense open set in G, Harish-Chandra constructed all the characters (these are

---

[1]) For compact groups this observation is due to Bott.

distributions on G) of the *discrete series* of representations of G, for G which has a compact Cartan subgroup H. By definition the latter representations are the irreducible subrepresentations of $L^2(G)$. Their existence, as shown by Harish-Chandra, coincides precisely with the existence in G of such a subgroup [26]. These discrete series representations are, in a very definite sense, the analogues (in the non-compact case) of the irreducible representations of compact groups. In fact by the Peter-Weyl theorem all of the irreducible representations of a compact group K occur in $L^2(K)$. Also the discrete series satisfy the "orthogonality relations" which for irreducible representations of compact groups are well-known.

The great importance of the discrete series representations is the following (which is in fact Harish-Chandra's guiding principle in his approach to the harmonic analysis on G): Every Cartan subgroup of G makes a specific contribution to the Plancherel formula which is determined by the discrete series of a suitable reductive group. The contributions of the compact Cartan subgroups (when they exist) is to the discrete part of the Plancherel formula.

Just as in the compact case, there naturally arises the question of realizing these discrete series representations (or even certain "continuous series"). It was Kostant and Langlands [51], [54] who first suggested that, in analogy with the Borel-Weil theorem, the discrete series representations should occur on L^2-cohomology spaces of homogeneous holomorphic line bundles over the complex manifold G/H. This important conjecture was recently confirmed by W. Schmid [82]. In our discussion of Schmid's results, we shall see (as in the compact case) a holomorphic version of Frobenius reciprocity.

Besides the work of Schmid, fundamental break-throughs towards the verification of the Kostant-Langlands conjecture appeared in the work of M. S. Narasimhan and K. Okamoto [60]. An alternative to the L^2-cohomology realization of the discrete series is the realization by means of invariant differential operators. In this regard very attractive results have been obtained in particular by R. Parthasarathy by way of the Dirac operator [65]. Recent analogues of the Kostant-Langlands conjecture for nilpotent Lie groups have been proposed and proved by H. Moscovici, A. Verona, J. Rosenberg and R. Penney [59], [74] [69]. For the realizations of non-discrete series representations of reductive groups on "partially holomorphic cohomology spaces" the reader may consult the important AMS Memoir of J. Wolf [99].

This lecture is dedicated to the memory of my undergraduate mathematics teacher and dear friend Dr. Walter R. Talbot.

2. Induction and reciprocity

The notion of induced representations for finite groups was introduced in 1898 by G. Frobenius in the paper [37]. In the same paper Frobenius established what is now called the Frobenius reciprocity relation. We recall his basic construction which is fundamental in the entire theory of group representations. [1])

Let G be a finite group and let P be a subgroup of G. Let π be a representation of G on a finite dimensional vector space V. That is $\pi : G \to GL(V)$ is a homomorphism of G into the group of non-singular endomorphisms of V. We shall also refer to V as a (left) G module. By restriction V is also a P module. Conversely there is a functor I which converts P modules to G modules: Given a P module W the G module IW is defined to be the space of functions $f : G \to W$ such that $f(ap) = p^{-1} \cdot f(a)$ for every (a, p) in $G \times P$. The action of G on IW is defined by

$$(a \cdot f)(x) = f(a^{-1}x)$$

for (f, a, x) in $(I\,W) \times G \times G$. $I\,W$ is called the G module *induced* by the P module W. Induction and restriction are related in the following way.

THEOREM 2.1 (Frobenius reciprocity relation, 1898). *If W is a P module and if V is a G module then*

$$\mathrm{Hom}_G(V, I\,W) = \mathrm{Hom}_P(V, W).$$

We wish to consider extensions or analogues of this relation in a wider context. For this it is most convenient first of all to re-describe the G module $I\,W$. The following "geometric" interpretation of $I\,W$ is well-known. Consider the right action of P on $G \times W$ given by

$$(a, w) \cdot p = (ap, p^{-1}w)$$

for (a, p, w) in $G \times P \times W$. Let

$$E_W = \text{orbit space } (G \times W)/P = G \times_P W. \qquad (2.2)$$

Let $\gamma : E_W \to G/P$ be the canonical (well-defined) map $[a, w] \to aP$, where $[a, w]$ is the orbit of $(a, w) \in G \times W$. For each $a \in G$ the map $w \to [a, w]$ of W to $\gamma^{-1}\{aP\}$ is a bijection. That is we may identify W as the fibre over each point of

---

[1]) For the theory of induced representations of locally compact groups see G. Mackey [55], [56].

G/P. G acts naturally on E_W and G/P on the left. γ is an equivariant map. Let $\Gamma(E_W)$ be the space of sections of E_W. That is $s \in \Gamma(E_W)$ is a map from G/P to E_W satisfying $\gamma \circ s = 1$; hence s maps each point to the fibre over it. $\Gamma(E_W)$ is a left G module:

$$(a \cdot s)(x) = a \cdot s(a^{-1} \cdot x) \tag{2.3}$$

for (a, s, x) in $G \times \Gamma(E_W) \times G/P$. Moreover

PROPOSITION 2.4. *There is a natural* G *module isomorphism* $s \to f^s$ *of* $\Gamma(E_W)$ *onto* $I\,W$ *such that for every* a *in* G, $s(aP) = [a, f^s(a)]$. *Hence by Theorem 2.1*

$$\mathrm{Hom}_G(V, \Gamma(E_W)) = \mathrm{Hom}_P(V, W). \tag{2.5}$$

This sets the stage for a possible extension of Frobenius. Namely, following Bott, we consider the following data. G is a complex Lie group, P is a closed complex Lie subgroup (thus the injection $P \to G$ is holomorphic), and W is a finite dimensional holomorphic P module (i.e. for each w in W and f in the complex dual space of W the map $p \to f(p \cdot w)$ of P to the complex numbers is holomorphic). We define E_W exactly as above. Then E_W has the structure of a holomorphic vector bundle over the complex manifold G/P. Let $\Gamma(E_W)$ now denote the space of C^∞ sections with the G module structure given by (2.3) and let $\Gamma_{\mathrm{hol}}(E_W)$ denote the G stable subspace of holomorphic sections. Since all of our data is now holomorphic the most natural question to ask, considering (2.5), is: When is it true that

$$\mathrm{Hom}_G(V, \Gamma_{\mathrm{hol}}(E_W)) = \mathrm{Hom}_P(V, W) \tag{2.6}$$

for a holomorphic G module V? (2.6) would then represent an exact holomorphic analogue of Frobenius reciprocity. It turns out that (2.6) is valid if the space G/P is sufficiently nice. For example suppose that G/P is a compact simply connected Kahler manifold. Group theoretically this means that G is a connected complex semisimple Lie group and P is a parabolic subgroup. Then it is due to Bott [12] that (2.6) is valid. In fact in [12] Bott proves considerably more: Let SE_W be the sheaf of germs of local holomorphic sections of E_W and let $H^*(G/P, SE_W)$ be the cohomology of G/P with coefficients in SE_W. Then we have

THEOREM 2.7 (R. Bott, 1957). *Suppose* G *is a connected complex semisimple Lie group and* P *is a parabolic subgroup of* G. *Let* $\mathfrak{p}$ *be the Lie algebra of* P *and let* V, W *be finite dimensional holomorphic* G *and* P *modules respectively. Then*

(2.8) $$\operatorname{Hom}_G(V, H^j(G/P, SE_W)) = H^j(p, p \cap \bar{p}, \operatorname{Hom}(V, W))$$

for each $j \geqslant 0$.

The bar – denotes conjugation of G with respect to a maximal compact subgroup K of G and the right hand side of (2.8) is the *relative* Lie algebra cohomology of p (in the sense of Hochschild, Serre [44]). Here $H^j(G/P, SE_W)$ [1]) has the G module structure induced by the left action of G on E_W and Hom (V, W) has the p module structure defined by

(2.9) $$(x \cdot \phi)(v) = -\phi(x \cdot v) + x \cdot \phi(v)$$

for (x, ϕ, v) in $p \times \operatorname{Hom}(V, W) \times V$.

Remarks. (i) For $j = 0$, $H^0(p, p \cap \bar{p}, \operatorname{Hom}(V, W))$ is independent of the subalgebra $p \cap \bar{p}$ of p and has the value $\operatorname{Hom}(V, W)^P$ (the space of invariants) which is precisely $\operatorname{Hom}_p(V, W) = \operatorname{Hom}_P(V, W)$ by (2.9) (P is connected). Also $H^0(G/P, SE_W)$ is precisely $\Gamma_{\text{hol}}(E_W)$. Thus taking $j = 0$ in (2.8) we get

$$\operatorname{Hom}_G(V, \Gamma_{\text{hol}}(E_W)) = \operatorname{Hom}_P(V, W)$$

which is (2.6). This shows that (2.8) represents a rather remarkable extension of Frobenius reciprocity to higher cohomology. Here the induction functor is $I: W \to H^*(G/P, SE_W)$.

(ii) As shown by Bott (2.8) is valid, more generally, for C-spaces G/P in the sense of Wang [90]. The latter need not be Kahler, as we have assumed for our purposes.

The functor I in remark (i) can be explicated by the use of differential forms: Let $\Lambda^{0,j}(G/P, E_W)$ denote the space of E_W valued C^∞ differential forms on G/P of pure type $(0, j)$. That is

$$\omega \in \Lambda^{0,j}(G/P, E_W)$$

assigns to each $x \in G/P$ a skew-symmetric j linear map

$$\omega_x : T_x(G/P)^{\mathbf{C}} \times \ldots \times T_x(G/P)^{\mathbf{C}} \to (E_W)_x = \gamma^{-1}\{x\}$$

on the complexified tangent space $T_x(G/P)^{\mathbf{C}}$ of G/P at x to the fiber $(E_W)_x$ over x such that (a) given smooth vector fields $X_1, \ldots, X_j$ on G/P the map

$$\omega(X_1, \ldots, X_j) : x \to \omega_x(X_{1_x}, \ldots, X_{j_x})$$

is C^∞—i.e. it belongs to $\Gamma(E_W)$ and (b) for each real number θ,

$$\omega(U_\theta X_1, \ldots, U_\theta X_j) = e^{-\sqrt{-1}\, j\theta}\, \omega(X_1, \ldots, X_j)$$

[1]) Since G/P is compact $H^j(G/P, SE_W)$ is known to be finite-dimensional.

where

$$U_\theta X_l = \cos\theta\, X_l + \sin\theta\, JX_l$$

and J is the complex structure tensor on G/P. Let $\bar\partial: \Lambda^{0,j} \to \Lambda^{0,j+1}$ denote, as usual, the Cauchy-Riemann operator so that $\bar\partial^2 = 0$. If f is a C^∞ function on G/P and X is a C^∞ vector field on G/P then

(2.10) $$(\bar\partial f)(X) = \frac{1}{2}[Xf + \sqrt{-1}\,(JX)f].$$

Since $\bar\partial^2 = 0$ let $H^{0,j}_{\bar\partial}(G/P, E_W)$ denote the corresponding $\bar\partial$ cohomology:

(2.11) $$H^{0,j}_{\bar\partial}(G/P, E_W) = \frac{\ker \bar\partial: \Lambda^{0,j}(G/P, E_W) \to \Lambda^{0,j+1}(G/P, E_W)}{\bar\partial\Lambda^{0,j-1}(G/P, E_W)}.$$

By Dolbeault's theorem [35]

(2.12) $$H^j(G/P, SE_W) = H^{0,j}_{\bar\partial}(G/P, E_W).$$

The induced action of G on $H^{0,j}_{\bar\partial}(G/P, E_W)$ is given explicitly as follows. First G acts on $\Lambda^{0,j}(G/P, E_W)$ by

(2.13) $$(a\cdot\omega)_x(L_1, \ldots, L_j) = a\cdot\omega_{a^{-1}x}\left(dl_{a^{-1}x}(L_1), \ldots, dl_{a^{-1}x}(L_j)\right)$$

where

$$(a, \omega, x) \in G \times \Lambda^{0,j}(G/P, E_W) \times G/P,$$

each $L_l \in T_x(G/P)^{\mathbf{C}}$ and dl_{a_x} is the derivative of left translation $l_a: G/P \to G/P$ on G/P at x. Note that (2.13) generalizes the action of G on

$$\Gamma(E_W) = \Lambda^{0,0}(G/P, E_W)$$

given in (2.3). Because left translation is holomorphic the diagram

$$\begin{array}{ccc} \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar\partial} & \Lambda^{0,j+1}(G/P, E_W) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar\partial} & \Lambda^{0,j+1}(G/P, E_W) \end{array}$$

is commutative for each a in G. Thus (2.13) induces a well-defined action of G on $H^{0,j}_{\bar\partial}(G/P, E_W)$. We may now write (2.8) as

(2.14) $$\mathrm{Hom}_G(V, H^{0,j}_{\bar\partial}(G/P, E_W)) = H^j(p, p\cap\bar p, \mathrm{Hom}(V, W)).$$

Now assume that W is in fact irreducible. The parabolic subalgebra p has a decomposition $p = (p \cap \bar{p}) \oplus n$ into a reductive part $p \cap \bar{p}$ and a nilpotent part $n =$ an ideal in p. By general principles

$$H^j(p, p \cap \bar{p}, \operatorname{Hom}(V, W)) = H^j(n, \operatorname{Hom}(V, W))^{p \cap \bar{p}}$$
$$= H^j(n, V^* \otimes W)^{p \cap \bar{p}} = \left(H^j(n\ V^*) \otimes W\right)^{p \cap \bar{p}}.$$

The last statement of equality follows by the irreducibility of W since by Lie's theorem, W is a trivial n module. Now

$$\left(H^j(n, V^*) \otimes W\right)^{p \cap \bar{p}} = \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V^*)\right).$$

From (2.14) we obtain (see [50]).

THEOREM 2.15 (Bott-Kostant reciprocity, 1960). *Let G, P be as in Theorem 2.7, let n be the nilradical of the parabolic subalgebra p, and let W be a finite dimensional irreducible holomorphic P module. Then for any finite dimensional holomorphic G module V we have*

$$\operatorname{Hom}_G\left(V, H^{0,j}_{\bar{\partial}}(G/P, E_W)\right) = \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V^*)\right). \tag{2.16}$$

Again $p \cap \bar{p}$ is the reductive part of p where the bar denotes conjugation of $G = K^{\mathbf{C}}$ with respect to a maximal compact subgroup K. We refer to (2.16) as "the debut of n cohomology"! Since 1960 it has played some rather important roles in both finite dimensional and infinite dimensional representation theory. There is an equivalent version of (2.16): The G module structure on $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ induced by (2.13) may be restricted to K. Let $\hat{K}$ denote, as usual, the equivalence classes of the irreducible unitary representations of K and let V_π be the representation space of $\pi \in \hat{K}$. Then we have (again for W irreducible).

THEOREM 2.17 (B. Kostant). *The decomposition of $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ as a K module is*

$$H^{0,j}_{\bar{\partial}}(G/P, E_W) = \sum_{\pi \in \hat{K}} V_\pi \otimes \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V_\pi^*)\right)$$
$$= \sum_{\pi \in \hat{K}} V_\pi^* \otimes \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V_\pi)\right). \tag{2.18}$$

In the direct sum on the right hand side the action of K on a summand is $\pi \otimes 1$ or $\pi^* \otimes 1$ in the second equation.

From (2.18) (or from (2.16)) we see that the multiplicity of an irreducible K module V_π in $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ is governed precisely by the n cohomology

$H^j(n, V_\pi^*)$. Here, by analytic continuation, we consider V_π also as a representation of the complex Lie algebra of G. Its n module structure is the restriction thereof to n.

Remarks. (i) In contrast to remark (ii) made earlier, following Theorem 2.7, Theorems (2.15) and (2.17) do require that G/P should be Kahler.

(ii) One knows that K acts transitively on G/P so that G/P is diffeomorphic to $K/K \cap P$.

Now Kostant in [50] has computed the Lie algebra cohomology groups $H^j(n, V_\pi^*)$. Two outstanding consequences of his results, among others, which we shall briefly discuss are (a) Weyl's character formula and (b) Bott's generalized Borel-Weil theorem. Suppose more generally that g is any complex semisimple Lie algebra (for example g could be the Lie algebra of G above). Let $h \subset g$ be a Cartan subalgebra of g, let Δ be the set of non-zero roots of (g, h), and let Δ^+ be a choice of positive roots. The equivalence classes of finite dimensional irreducible representations of g (over the complex numbers) correspond univalently to linear functionals Λ on h which satisfy the condition that $2\frac{(\Lambda, \alpha)}{(\alpha, \alpha)}$ is a non-negative integer for each α in Δ^+. That is Λ is Δ^+ *dominant integral*; $(\, , \,)$ denotes the Killing form on g. This is Cartan's highest weight theory alluded to in the introduction. Let π_Δ be a finite dimensional irreducible representation of g with corresponding highest weight $\Lambda \in h^*$. Its *character* $X_\Lambda : h \to \mathbf{C}$ is defined to be the function $H \to \text{trace} \exp \pi_\Lambda(H)$, $H \in h$. This definition is independent of the choice of Cartan subalgebra h since any two are conjugate. We consider the special "minimal" parabolic subalgebra $p \subset g$ whose nilradical is

$$n = \sum_{\alpha \in \Delta^+} g_\alpha \tag{2.19}$$

and whose reductive part is h where g_α is the root space of $\alpha \in \Delta$. That is p is just the Borel subalgebra $h + n$. Let V_Λ denote the representation space of π_Λ. Then by restriction to n we again form the Lie algebra cohomology groups $H^j(n, V_\Lambda)$. Let θ denote the adjoint representation of h on Λn^*. Then $\theta \otimes \pi_\Lambda$ defines a representation of h on the cochain complex $\Lambda n^* \otimes V_\Lambda$. This h action commutes with the coboundary operator and therefore passes to cohomology. Applying the Euler-Poincaré principle one gets

$$\sum_{j=0}^{\dim n} (-1)^j \, \text{trace} \exp \theta \otimes \pi_\Lambda(H) \Big|_{\Lambda^j n^* \otimes V_\Lambda} = \sum_{j=0}^{\dim n} (-1)^j \, \text{trace} \exp \theta \otimes \pi_\Lambda(H) \Big|_{H^j(n, V_\Lambda)} \tag{2.20}$$

for each H in h. One evaluates the left hand side of (2.20) by general principles and the right hand side using Kostant's main theorem, Theorem 5.14 of [50]. Actually Theorem 5.14 of [50] gives the h_1 module structure of $H^j(n_1, V_\Lambda)$ for an *arbitrary* parabolic $p_1 = h_1 + n_1$ of g with reductive and nilpotent parts h_1, n_1 respectively. For the derivation of Weyl's formula only the simplest case $p_1 = p = h + n$ is needed, where n is given in (2.19). Thus we shall state only a special case of Kostant's result.

THEOREM 2.21 (B. Kostant, 1960). *The decomposition of* $H^j(n, V_\Lambda)$ *as a* h *module is*

$$H^j(n, V_\Lambda) = \sum V_{\Lambda,\sigma},$$

$$\sigma \in \text{Weyl group } \mathscr{W} \text{ of } (g, h) \text{ such that } l(\sigma) = j,$$

where each summand $V_{\Lambda,\sigma}$ *in the direct sum is one-dimensional and* $H \in h$ *acts on* $V_{\Lambda,\sigma}$ *by the scalar* $[\sigma(\Lambda+\delta)-\delta](H)$.

Here by definition $2\delta = \sum_{\alpha \in \Delta^+} \alpha$ and $l(\sigma)$ (the *length* of σ) is the cardinality of the set $\Delta^+ \cap \sigma(-\Delta^+)$. From the remarks following (2.20) and the knowledge of n cohomology given by Theorem 2.21 one derives Weyl's famous character formula [93]:

THEOREM 2.22 (H. Weyl, 1926). *For* $H \in h$

$$X_\Lambda(H) = \frac{\sum_{\sigma \in \mathscr{W}} (\det \sigma)\, e^{[\sigma(\Lambda+\delta)](H)}}{\prod_{\alpha \in \Delta^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}.$$

The denominator is also given by the sum $\sum_{\sigma \in \mathscr{W}} (\det \sigma)\, e^{(\sigma\delta)(H)}$ (this fact can be proved too using n cohomology) and $\det \sigma = (-1)^{l(\sigma)}$. As a corollary of Theorem 2.22 one obtains Weyl's formula for the dimension of the irreducible module V_Λ in terms of its highest weight Λ. The result is

$$\text{(2.23)} \qquad \dim V_\Lambda = \frac{\prod_{\alpha \in \Delta^+} (\Lambda+\delta, \alpha)}{\prod_{\alpha \in \Delta^+} (\delta, \alpha)}.$$

Kostant's result on n cohomology can also be used to derive the generalized Borel-Weil theorem. Here one may apply formula (2.18) decisively. Let g now denote the Lie algebra of G. Extend a maximal abelian subalgebra of the Lie algebra of K to a Cartan subalgebra h of g. Again let $\Delta^+ \subset \Delta$ be a choice of positive roots where Δ is the set of non-zero roots of (g, h) and let $2\delta = \sum_{\alpha \in \Delta^+} \alpha$.

We choose the parabolic P such that its Lie algebra p contains the Borel subalgebra $h + \sum_{\alpha \in \Delta^+} g_{-\alpha} \cdot h$ is also a Cartan subalgebra of the reductive Lie algebra $p \cap \bar{p}$ so that we have the decompositions

$$p = (p \cap \bar{p}) \oplus n, \qquad p \cap \bar{p} = h + \sum_{\alpha \in \Delta(p \cap \bar{p})} g_\alpha$$

(2.24)

$$n = \sum_{\alpha \in \Delta^+ - \Delta(p \cap \bar{p})} g_{-\alpha}$$

where $\Delta(p \cap \bar{p})$ is the set of roots of $(p \cap \bar{p}, h)$.

Let W be an irreducible holomorphic P module. Then W is an irreducible $p \cap \bar{p}$ module thereby such that $n \cdot W = 0$. We let Λ denote its highest weight relative to the positive system $\Delta^+ \cap \Delta(p \cap \bar{p})$ for $p \cap \bar{p}$. Applying Kostant's n cohomology theorem to (2.18) one obtains (see [12], [50]).

THEOREM 2.25 (R. Bott, 1957). *The spaces* $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ *vanish for all but at most one* j. *If*

$$H^{0,j_0}_{\bar{\partial}}(G/P, E_W) \neq 0$$

then $H^{0,j_0}_{\bar{\partial}}(G/P, E_W)$ *is an irreducible* K *module.*

More precisely we have the following. Let Λ be the highest weight of W (as above) relative to the positive roots in the reductive part of P. If $(\Lambda+\delta, \alpha) = 0$ for some α in Δ then $H^{0,j}_{\bar{\partial}}(G/P, E_W) = 0$ for every j. If $(\Lambda+\delta, \alpha) \neq 0$ for each α in Δ (i.e. $\Lambda+\delta$ is *regular*) there is a unique element σ in the Weyl group of (g, h) such that $(\sigma(\Lambda+\delta), \alpha) > 0$ for every $\alpha \in \Delta^+$. Then $H^{0,j}_{\bar{\partial}}(G/P, E_W) = 0$ for $j \neq l(\sigma)$ where again $l(\sigma)$ is the length of σ (see remarks following Theorem 2.21). Moreover $H^{0,l(\sigma)}_{\bar{\partial}}(G/P, E_W)$ is an irreducible K module (= an irreducible g module since g is the complexification of the Lie algebra of K) with highest weight $\sigma(\Lambda+\delta) - \delta$ relative to Δ^+.

Remarks. (i) By definition of σ it follows that

$$\sigma^{-1}\Delta^- \cap \Delta^+ = \{\alpha \in \Delta^+ \mid (\Lambda+\delta, \alpha) < 0\}.$$

Also since Λ is a highest weight $(\Lambda, \alpha) \geqslant 0$ for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}) \Rightarrow (\Lambda+\delta, \alpha) > 0$$

for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}).$$

Hence

$$\{\alpha \in \Delta^+ \mid (\Lambda+\delta, \alpha) < 0\}$$
$$= \{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p})) \mid (\Lambda+\delta, \alpha) < 0\}$$

so that $l(\sigma)$ in Theorem 2.25 has the value

$$|\{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p})) \mid (\Lambda + \delta, \alpha) < 0\}|\ ^{1)}.$$

$$\Delta^+ - \Delta^+ \cap \Delta(p \cap \bar{p})$$

is the set of roots in the nilradical of the "opposite" parabolic $\bar{p}$. Since

$$(\sigma(\Lambda + \delta), \sigma\alpha) = (\Lambda + \delta, \alpha) > 0$$

for $\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p})$ (as we have just seen) we also conclude that the Weyl group element σ in Theorem 2.25 satisfies

$$\Delta^- \cap \Delta(p \cap \bar{p}) \subset \sigma^{-1} \Delta^- .$$

(ii) The irreducible holomorphic P modules W in the statement of Theorem 2.25 can be obtained as follows. Start with an arbitrary irreducible representation π of $P \cap K$ on a complex vector space W. Since $p \cap \bar{p}$ is the complexification of the Lie algebra of $P \cap K$, π defines a unique irreducible representation π on p such that $\pi(n) = 0$. This infinitesimal representation can be "integrated" to a representation of P since P and $P \cap K$ have the same fundamental groups. Thus every irreducible representation π of $P \cap K$ extends uniquely to an irreducible holomorphic representation of P. The highest weight Λ of π is integral and $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant. Conversely if G is simply connected, any integral $\Lambda \in h^*$ which is $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant is the highest weight of irreducible representation of $P \cap K$ and hence is the highest weight of an irreducible holomorphic representation of P.

(iii) Suppose in particular G is simply connected, p is chosen to be

$$h + \sum_{\alpha \in \Delta^+} g_{-\alpha},$$

and that Λ is Δ^+ dominant integral. Then in Theorem 2.25 $\sigma = 1$ so that the irreducible K, G or g module with highest weight Λ is given by $H^{0,0}_{\bar{\partial}}(G/P, E_W)$ = space of holomorphic sections of the line bundle E_W. Indeed $\dim_{\mathbf{C}} W = 1$ since in this case $P \cap K$ is abelian. This gives the geometric realization of V_Λ [11].

[1]) $|S|$ denotes the cardinality of a set S.

3. Representations of non-compact semisimple groups

In section 2 we interpreted the equation

$$(3.1) \qquad \mathrm{Hom}_G\left(V, H_{\bar\partial}^{0,j}(G/P, E_W)\right) = \mathrm{Hom}_{p \cap \bar p}\left(W^*, H^j(n, V^*)\right)$$

in (2.16) as a precise extension of Frobenius reciprocity to higher cohomology; also see (2.8), (2.12), and (2.14). There we used $\bar\partial$ cohomology and assumed that our space $G/P = K/K \cap P$ was compact. In this section we shall see yet another extension of Frobenius reciprocity where K is replaced by a real non-compact semisimple group and $\bar\partial$ cohomology is replaced by "square integrable" $\bar\partial$ cohomology. In this non-compact context the *discrete series* defined in the introduction will play the analogous role of the dual objects $\hat K$ of equivalence classes of irreducible unitary representations of K. The analogue of the generalized Borel-Weil theorem (Theorem 2.25) for example will be formulated for the discrete series. This is Schmid's solution of the Kostant-Langlands conjecture.

In this section G will now denote a real non-compact connected semisimple Lie group with finite center and K will denote a maximal compact subgroup of G. If π is any unitary representation of G on a Hilbert space H and $f \in L_1(G)$ we let

$$(3.2) \qquad \pi(f) = \int_G f(x)\, \pi(x)\, dx$$

so that $\pi(f)$ is a bounded operator on H satisfying $\pi(f * g) = \pi(f)\,\pi(g)$ for f, $g \in L_1(G)$. $*$ denotes convolution and dx denotes Haar measure on the unimodular group G. The following fundamental theorem of Harish-Chandra is valid ([20]).

Theorem 3.3. *If π is an irreducible unitary representation of G and $f \in C_c^\infty(G)$ is a compactly supported smooth function on G then the operator $\pi(f)$ is of trace class. Moreover the equation $\Theta_\pi(f) = \mathrm{trace}\, \pi(f)$, $f \in C_c^\infty(G)$, defines a distribution Θ_π on G in the sense of L. Schwartz. Θ_π depends only on the unitary equivalence class $[\pi]$ of π. For two such classes $[\pi_1], [\pi_2]$ we have $\Theta_{[\pi_1]} = \Theta_{[\pi_2]}$ if and only if $[\pi_1] = [\pi_2]$.*

The distribution Θ_π is called the (global) *character* of π (or of $[\pi]$). The fact that $\pi(f)$ is of trace class is a consequence of the following fundamental deep fact: There is an integer $N \geqslant 1$ such that for any irreducible unitary representations π, σ of G and K we have

$$\text{(3.4)} \qquad \text{the multiplicity of } \sigma \text{ in } \pi\Big|_K \leqslant N \text{ dimension of } \sigma\,.$$

Θ_π is invariant under all inner automorphisms of G and if $\mathscr{Z}$ is the algebra of bi-invariant differential operators [1]) on G then the space of distributions $\{Z\,\Theta_\pi \mid Z \in \mathscr{Z}\}$ has dimension one. That is Θ_π is $\mathscr{Z}$-finite and is in fact an *eigendistribution* of $\mathscr{Z}$. By Harish-Chandra's profound regularity theorem for invariant $\mathscr{Z}$-finite distributions [24] one may conclude that Θ_π is a locally integrable function on G which is actually analytic on the regular points G' of G. G' is an open dense set in G such that the complement $G - G'$ has Haar measure zero.

We now recall Harish-Chandra's character formula for the discrete series. We assume that G admits a Cartan subgroup H such that $H \subset K$. From the introductory remarks we recall that this assumption guarantees precisely that G has a discrete series. In order to avoid certain technical difficulties we shall assume moreover for simplicity that G is linear and that its complexification $G^{\mathbf{C}}$ is simply connected. Let g, k, h denote the complexifications of the Lie algebras g_0, k_0, h_0 of G, K, H respectively. As in section 2 we let Δ denote the set of non-zero roots of (g, h) and we let $2\delta = \sum_{\alpha \in \Delta^+} \alpha$ for a choice of positive system of roots $\Delta^+ \subset \Delta$. Again we say that $\Lambda \in h^*$ is *integral* if $2\dfrac{(\Lambda, \alpha)}{(\alpha, \alpha)}$ is an integer for each α in Δ. Thus $\exp X \to e^{\Lambda(x)}$ is a well defined character of H, $x \in h_0$.

THEOREM 3.5 (Harish-Chandra, 1964). *Let* $\Lambda \in h^*$ *be integral and suppose that* $(\Lambda + \delta, \alpha) \neq 0$ *for every* α *in* Δ [2]). *Then there exists a discrete series representation* π_Λ *of* G *such that*

$$\text{(3.6)} \qquad \Theta_{\pi_\Lambda}(\exp X) = \frac{(-1)^m \operatorname{sgn} \prod_{\alpha \in \Delta^+} (\Lambda + \delta, \alpha) \sum_{\sigma \in W_K} \det \sigma\, e^{[\sigma(\Lambda + \delta)](x)}}{\prod_{\alpha \in \Delta^+} \left(e^{\alpha(x)/2} - e^{-\alpha(x)/2}\right)}$$

---

[1]) We may identify $\mathscr{Z}$ with the center of the universal enveloping algebra of the complexified Lie algebra of G.

[2]) That is $\Lambda + \delta$ is regular.

for $x \in h_0$, where $2m = \dim G/K$ and W_K is the Weyl group of (K, H). Moreover every discrete series representation is equivalent to some π_Λ where $\Lambda \in h^*$ is integral and $\Lambda + \delta$ is regular and π_{Λ_1} is equivalent to π_{Λ_2} if and only if $\sigma(\Lambda_1 + \delta) = \Lambda_2 + \delta$ for some σ in W_K (see [26]).

The factor $(-1)^m \operatorname{sgn} \prod_{\alpha \in \Delta^+} (\Lambda + \delta, \alpha)$ in (3.6) can be expressed in an alternate form: Let Δ_n, Δ_k denote the set of non-compact and compact roots respectively. Thus by definition $\alpha \in \Delta_n$ (or Δ_k) if $g_\alpha \subset p_0^{\mathbf{C}}$ (or if $g_\alpha \subset k_0^{\mathbf{C}} = k$) where

(3.7) $$g_0 = k_0 + p_0$$

is a Cartan decomposition of g_0. Let $\Delta_n^+ = \Delta^+ \cap \Delta_n$ and $\Delta_k^+ = \Delta^+ \cap \Delta_k$. Then it is easy to check that

(3.8) $$(-1)^m \operatorname{sgn} \prod_{\alpha \in \Delta^+} (\Lambda + \delta, \alpha) = (-1)^{q_\Lambda}$$

where

(3.9) $$q_\Lambda \overset{\text{def.}}{=} |\{\alpha \in \Delta_n^+ \mid (\Lambda + \delta, \alpha) > 0\}| + |\{\alpha \in \Delta_k^+ \mid (\Lambda + \delta, \alpha) < 0\}|$$

and $|S|$ is the cardinality of a set S. One notes the similarity in appearance of Harish-Chandra's character formula (3.6) and Weyl's character formula of Theorem 2.22.

For an integral $\Lambda \in h^*$ such that $\Lambda + \delta$ is regular we continue to denote the corresponding discrete series representation of Theorem 3.5 by π_Λ. Let $\mathscr{L}_\Lambda \to G/H$ denote the C^∞ line bundle over G/H induced by the character (which we have seen is well-defined) $\exp x \to e^{\Lambda(x)}$ of H, $x \in h_0$. Let P denote the Borel subgroup of $G^{\mathbf{C}}$ corresponding to the subalgebra $p = h + \sum_{\alpha \in \Delta^+} g_{-\alpha}$ of g. Then $G \cap P = H$ so that by general principles, see [40], [78], G/H is an open G orbit in $G^{\mathbf{C}}/P$ and thus G/H has a G invariant complex structure such that, at the origin, $n = \sum_{\alpha \in \Delta^+} g_{-\alpha}$ is the space of anti-holomorphic tangent vectors; and moreover $\mathscr{L}_\Lambda$ also has a holomorphic structure [1]). G/H may therefore be considered as a non-compact analogue of the space $K/K \cap P$ of section 2 with $\mathscr{L}_\Lambda$ playing the analogous role of E_Λ. Thus given the Borel-Weil theorem (Theorem 2.25) one naturally inquires whether the representation π_Λ occurs on a $\bar{\partial}$ cohomology space of differential forms with coefficients in $\mathscr{L}_\Lambda$. This question was posed (more precisely) first by B. Kostant and R. Langlands in 1965. Since G/H is non-compact we should consider L_2-cohomology. Namely we proceed as

---

[1]) The G invariant complex structures on G/H correspond to the choices of positive root systems Δ^+.

follows. Let $\Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)$ denote the space of complex smooth compactly supported forms of type $(0, j)$ on G/H with values in $\mathscr{L}_\Lambda$; cf. remarks preceding (2.10). With respect to natural G invariant hermitian metrics on the fibres of $\mathscr{L}_\Lambda$ and on the tangent bundle of G/H $\Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)$ has the structure of a complex inner product space and the Cauchy-Riemann operator

$$\bar{\partial} : \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda) \to \Lambda_c^{0,j+1}(G/H, \mathscr{L}_\Lambda)$$

has a formal adjoint

$$\bar{\partial}^* : \Lambda_c^{0,j+1}(G/H, \mathscr{L}_\Lambda) \to \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda) .$$

As usual let

$$\begin{aligned} \square = \bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\,\bar{\partial} : \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda) \\ \to \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda) \end{aligned}$$

denote the corresponding complex Laplace-Beltrami operator. Let $L_2^{0,j}(G/H, \mathscr{L}_\Lambda)$ denote the Hilbert space completion of $\Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)$ and set

(3.10)
$$\begin{aligned} &H_{\bar{\partial}, 2}^{0,j}(G/H, \mathscr{L}_\Lambda) \\ &= \{\phi \in L_2^{0,j}(G/H, \mathscr{L}_\Lambda) \mid \square\, \phi = 0 \text{ in the distribution sense}\} . \end{aligned}$$

$H_{\bar{\partial}, 2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ is called the L_2-cohomology (or harmonic) space attached to the Hermitian line bundle $\mathscr{L}_\Lambda$. The reader should consult [63] for a more general definition of L_2-cohomology spaces [1]. We take this opportunity to point out that in [63] line 9 of page 96 should be corrected to read $\vartheta = (\bar{\partial})^*$ rather than $\vartheta = (\bar{\partial}_0)^*$.

One knows that $H_{\bar{\partial}, 2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ is a closed subspace of

$$L_2^{0,j}(G/H, \mathscr{L}_\Lambda)$$

and hence it is a Hilbert space. Moreover since the above hermitian metrics were chosen to be G invariant, $H_{\bar{\partial}, 2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ carries a natural unitary representation of G-namely that induced by left translation on forms; cf. (2.13). We denote this representation by $\pi_\Lambda^{0,j}$. The analogies with section 2 presist since in the compact case classical Hodge theory represents the $\bar{\partial}$ cohomology in (2.11) as solutions of a complex Laplacian . □

Now Kostant and Langlands conjectured, in analogy with Theorem 2.25, that the spaces $H_{\bar{\partial}, 2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ should vanish for each j if $\Lambda + \delta$ is not regular and if $\Lambda + \delta$ is regular (as we have assumed) then one should have vanishing for

---

[1]) The definition (3.10) suffices for our purposes since we are dealing with a homogeneous space G/H where the metric is therefore automatically complete.

all but one j, say $j = j_0$. Moreover the representations π_Λ^{0, j_0} and π_Λ should coincide; see [51], [54]. Based on a vanishing theorem of P. Griffiths [40] some results of Harish-Chandra [22] and a formal application of the Wood's Hole fixed point formula [54], Langlands predicted moreover that the value of j_0 should be the number q_Λ given in (3.9). Important progress towards the verification of the Kostant-Langlands conjecture was made by W. Schmid in his 1967 thesis [77] and by Schmid [78], M. S. Narasimhan, and K. Okamoto in 1969 [60]. In 1973 W. Casselman and M. Osborne proved a version of Kostant's theorem on n cohomology (see Theorem 2.21) in the case where the coefficient module (for g) is *infinite dimensional* but has an infinitesimal character [17]. Schmid used the Casselman-Osborne result decisively in [82] and thereby settled the conjecture:

THEOREM 3.11 (W. Schmid, 1975). *Let* $\Lambda \in h^*$ *be an integral element. If* $(\Lambda + \delta, \alpha) = 0$ *for some root* α *then the space*

$$H^{0, j}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda)$$

in (3.10) vanishes for each j. *Suppose that* $\Lambda + \delta$ *is regular. Let* π_Λ *be the corresponding discrete series representation whose character is given by (3.6) in Theorem 3.5. Let* q_Λ *be the integer defined in (3.9). Then*

$$H^{0, j}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda) = 0$$

for $j \neq q_\Lambda$ *and the natural unitary representation* $\pi_\Lambda^{0, q_\Lambda}$ *of* G *on*

$$H^{0, q_\Lambda}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda)$$

is irreducible and equivalent to π_Λ.

Remarks. In the preceding we have assumed for simplicity that G was linear. It is now known that this assumption can be dropped in the statement of Theorem 3.11. We may also drop the integrality of Λ in the statement of Theorem 3.11 and assume, more generally, only that Λ is real-valued on roots and extends to a character of H. It is then still true that $\mathscr{L}_\Lambda$ carries a holomorphic structure; see remark 3 in [78].

The point we wish to stress now is that just as Theorem 2.25 is derivable from the Frobenius reciprocity relation (3.1), or (equivalently) from (2.18), Theorem 3.11 is like-wise derivable from a non-compact analogue of (2.18). In [78] (lemma 6) Schmid obtains a direct integral decomposition of the L^2-harmonic spaces $H^{0, j}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda)$ using the Plancherel decomposition of $L^2(G)$:

(3.12) $$H^{0,j}_{\bar{\partial},2}(G/H, \mathscr{L}_\Lambda) = \int_{\hat{G}} V^*_\pi \otimes \mathscr{H}^j(\pi)_{e^{-\Lambda}}\, d\pi$$

where $\mathscr{H}^j(\pi)$ is a certain *formal harmonic space* attached to the irreducible unitary representation $\pi \in \hat{G}$ of G and $\mathscr{H}^j(\pi)_{e^{-\Lambda}}$ is the subspace of $\mathscr{H}^j(\pi)$ transforming under the action of H according to the character $e^{-\Lambda}$. The tensor product in (3.12) is a tensor product of Hilbert spaces; V^*_π is the contragradient representation space of $(\pi, V_\pi) \in \hat{G}$ (where π acts on the Hilbert space V_π). Given $(\pi, V_\pi) \in \hat{G}$, $\mathscr{H}^j(\pi)$ is defined as follows. First let V^∞_π denote the space of K finite vectors in V_π (these are just the vectors in V_π whose K translates span a finite-dimensional subspace). In the usual way, via differentiation, V^∞_π is a g module (the induced action is skew-hermitian) and by restriction (as in section 2) V^∞_π is an n module where we take

(3.13) $$n = \sum_{\alpha \in \Delta^+} g_{-\alpha}$$

n has a natural Ad_H invariant inner product induced by the Killing form of g; see equation (2.4) of [78]. Thus we may consider the formal adjoint δ^* of the Lie algebra coboundary operator

$$\delta: V^\infty_\pi \otimes \Lambda n^* \to V^\infty_\pi \otimes \Lambda n^*$$

corresponding to the n module V^∞_π. One has

THEOREM 3.14 (Lemma 3 of [78]). $\delta + \delta^*$ *has a unique self-adjoint extension* $\overline{\delta + \delta^*}$. *Also* $\overline{\delta + \delta^*}$ *is the only closed extension of* $\delta + \delta^*$.

$V^\infty_\pi \otimes \Lambda^j n^*$ is dense in $V_\pi \otimes \Lambda^j n^*$ for each j (since V^∞_π is dense in V_π). By definition $\mathscr{H}^j(\pi)$ is the kernel of $\overline{\delta + \delta^*}$ considered as a closed densely defined operator in $V_\pi \otimes \Lambda^j n^*$. The H action on the cochains $V^\infty_\pi \otimes \Lambda^j n^*$ commutes with δ and also with δ^* (since the n inner product is Ad_H invariant) which means that $\mathscr{H}^j(\pi)$ (and the Lie algebra cohomology $H^j(n, V^\infty_\pi)$) inherits an H module structure. The subspace $\mathscr{H}^j(\pi)_{e^{-\Lambda}}$ in (3.12) of vectors in $\mathscr{H}^j(\pi)$ transforming according to the character $e^{-\Lambda}$ under this H action is therefore well-defined.

In [82] (Theorem 3.1) Schmid proves the H module isomorphism

(3.15) $$\mathscr{H}^j(\pi) \simeq H^j(n, V^\infty_\pi)$$

From the Casselman-Osborn result [17] (which as we have pointed out is a version of Kostant's Theorem 2.21 for infinite dimensional g modules with an infinitesimal character $- V^\infty_\pi$ being such an example),

$$H^j(n, V^\infty_\pi)_{e^\mu} = 0$$

unless V_π^∞ has a specific infinitesimal character (this means that on V_π^∞ the center of the universal enveloping algebra Ug of g must act by a specific scalar). In Harish-Chandra's notation [20] this character is $\chi_{-\mu-\delta}$, where again $2\delta = \sum_{\alpha \in \Delta^+} \alpha$; here $\mu \in h^*$ is integral or, more generally, μ is real-valued on roots and defines a character of H (see remarks following Theorem 3.11). On the other hand, from the harmonic analysis of G it is known that only finitely many irreducible unitary equivalence classes can have a fixed infinitesimal character and that moreover if $F \subset \hat{G}$ is a finite set which is disjoint from the classes of discrete series, then the Plancherel measure must vanish on F. Thus from these observations one concludes from (3.15) that only discrete series modules (π, V_π) can occur in the direct integral decomposition given in (3.12) and since the (π, V_π) occur discretely we obtain (cf. Corollary 3.23 of [82]) the following refinement of (3.12).

THEOREM 3.16 (Frobenius-Schmid reciprocity, 1975). *As G modules*

$$H^{0,j}_{\bar{\partial},2}(G/H, \mathscr{L}_\Lambda) = \sum_{\substack{(\pi, V) = \\ \text{discrete class}}} V_\pi^* \otimes H^j(n, V_\pi^\infty)_{e^{-\Lambda}}$$

This is the non-compact analogue of (2.18) (where the contragradient W^* of the inducing module W there is replaced by the contragradient $e^{-\Lambda}$ of the inducing character e^Λ). Theorem 3.16 preceeds and implies (with the knowledge of n cohomology, as in the compact case) Theorem 3.11.

4. REMARKS ON THE NILPOTENT CASE: POLARIZATIONS AND HARMONIC INDUCTION

The Frobenius reciprocity in higher cohomology discussed in the two preceding sections extends to a non-semisimple Lie group context as well. Moreover consequent analogues of the Kostant-Langlands conjecture have been proved. Most recently (within the past few months) remarkable and complete results along these lines have been obtained (independently) for simply connected nilpotent Lie groups by J. Rosenberg [74] and R. Penney [69]. Their results are preceded by results of H. Moscovici and A. Verona [59]; also see [15], [58], [62], [67], [68], [75]. In this regard one of the central notions to consider is that of a *polarization*. It is defined as follows. Let g be a real Lie algebra, let $\Lambda \in g^*$

be a linear functional, and let B_Λ be the skew symmetric bilinear form on g defined by

$$B_\Lambda(x, y) = \Lambda([x, y]), \; x, y \in g .$$

A (complex) polarization of g at Λ is a complex subalgebra p of $g^{\mathbf{C}}$ which is maximally isotropic relative to B_Λ and which has the further property that $p + \bar{p}$ is also a complex subalgebra of $g^{\mathbf{C}}$. The bar denotes conjugation of $g^{\mathbf{C}}$ relative to the real form g. Λ defines a hermitian sequilinear form H_Λ on

$$p: H_\Lambda(x, y) = \sqrt{-1}\,\Lambda([x, \bar{y}]) = \sqrt{-1}\,B_\Lambda(x, \bar{y}), \; x, y \in p .$$

Define

(4.1) $$q(p, \Lambda) = \dim_{\mathbf{C}} \frac{(p \cap \bar{p})}{\text{radical of } B_\Lambda} + \text{number of negative signs in the signature of } H_\Lambda \text{ on } \frac{p}{p \cap \bar{p}} .$$

This important invariant of the polarization is called its *negativity index*.

Let G now denote a connected, simply connected nilpotent Lie group with Lie algebra g and let p be a complex polarization of g at $\Lambda \in g^*$. Let d and e denote the subalgebras of g defined by

$$d = p \cap g, \; e = (p + \bar{p}) \cap g ;$$

hence

$$d^{\mathbf{C}} = p \cap \bar{p}, \; e^{\mathbf{C}} = p + \bar{p} .$$

Let $D, E \subset G$ be the corresponding connected Lie subgroups of G. One knows that D and E are closed (and simply connected) in G and the quotient $X = E \backslash D$ has a unique E invariant complex structure such that $p/d^{\mathbf{C}}$ is the anti-holomorphic tangent space at the origin. This is proved by Kostant in [53] for example. However X may not necessarily admit an E invariant hermitian metric. A sufficient condition for the existence of the latter is that $p \cap \bar{p}$ should be an ideal in p (for then the image of Ad (D) in Hom (e/d) is compact). Polarizations satisfying this sufficient condition are called *relatively ideal*. Now let X_Λ be the unique unitary character of D with differential $2\pi\sqrt{-1}\,\Lambda|_d$ and let $\mathscr{L}_\Lambda \to X$ be the corresponding induced C^∞ line bundle over X. Then a priori $\mathscr{L}_\Lambda$ admits an E invariant hermitian metric and a holomorphic structure. Thus if p is relatively ideal, which we now assume, so that X also admits an E invariant hermitian metric, the pair $(\mathscr{L}_\Lambda, X)$ is a hermitian bundle and the corresponding L^2-

cohomology groups $H^{0,j}_{\bar\partial,2,p,E}(X, \mathscr{L}_\Lambda)$ can therefore be defined; see 3.10 and [63]. This time we denote their dependence on p. Since the above metrics are E invariant these groups (= Hilbert spaces) carry a natural unitary representation of E (as in section 3) which we denote by $\pi^{0,j}_{\Lambda,p,E}$. Now form the induced representation of G in the sense of G. Mackey [55]:

(4.2)
$$\pi^j(\Lambda, p, G) \overset{def}{=} \underset{E \uparrow G}{\text{ind}} \; \pi^{0,j}_{\Lambda,p,E}$$

$\pi^j(\Lambda, p, G)$ is the j-th *harmonically* induced representation of G associated to the polarization p at Λ in the sense of Moscovici and Verona [59].

Now Theorem 3.5 of section 3 gives the "Harish-Chandra correspondence" $\Lambda_1 \to \pi_{\Lambda_1}$ where Λ_1 is an integral linear form on a Cartan subalgebra such that $\Lambda_1 + \delta$ is regular and π_{Λ_1} is the corresponding discrete series representation. Similarly there is for connected, simply connected nilpotent Lie groups G_1 the well known *Kirillov correspondence* $\Lambda_1 \to \pi_{\Lambda_1}$ [1]) $\in \hat{G}_1$ for $\Lambda_1 \in$ dual space of the Lie algebra g_1 of G_1, where in fact the whole unitary dual space $\hat{G}_1$ of G_1 is parametrized by the orbits in g_1^* under the contragredient action of the adjoint representation of G_1; see [49], [9], [72]. In terms of the Kirillov correspondence and harmonically induced representations we shall discuss another version of Frobenius reciprocity.

Recall the *formal harmonic spaces* $\mathscr{H}^j(\pi)$ of Schmid which appeared in equation (3.12) and defined thereafter. One may define similarly the j-th formal harmonic spaces $\mathscr{H}^j(\pi, p)$ of $\pi \in \hat{E}$ associated to the polarization p; see page 67 of [59]. In Lemma 4 of [59], or Theorem 10 of [68], Moscovici, Verona, and Penney prove

THEOREM 4.3 (1978). *Let p be a relatively ideal polarization at $\Lambda \in g^*$ as above and let $\pi_{\pm\Lambda|e} \in \hat{E}$ be the Kirillov representations of E corresponding to $\Lambda|_e \in e^*$. Then there is a Hilbert space isomorphism*

(4.4)
$$H^{0,j}_{\bar\partial,2,p,E}(X, \mathscr{L}_\Lambda) = (\text{representation space of } \pi_{\Lambda|e}) \otimes \mathscr{H}^j(\pi_{-\Lambda|e}, p)$$

such that $\pi^{0,j}_{\Lambda,p,E} = \pi_{\Lambda|e} \otimes 1$.

In other words the multiplicity of the Kirillov representation $\pi_{\Lambda|e}$ in the representation $\pi^{0,j}_{\Lambda,p,E}$ on $\bar\partial$-cohomology is given by the dimension of the formal harmonic space $\mathscr{H}^j(\pi|_{-\Lambda|e}, p)$. This result is rather similar to Theorem 3.16

[1]) π_{Λ_1} is the representation of G_1 induced by a unitary character corresponding to Λ_1 of a closed subgroup corresponding to a *real* polarization at Λ_1.

since by (3.15) the Lie algebra cohomology space in Theorem 3.16 is a formal harmonic space. It is even true as a matter of fact that under reasonable conditions the formal harmonic space associated to a polarization coincides with a Lie algebra cohomology space; see Penney's Theorem 2 in [68]. The latter cohomology spaces have the form $H^j(p \cap \operatorname{Ker} \Lambda, \pi^\infty_{-\Lambda})$ where π^∞ is the space of C^∞ vectors in a representation π and Λ is considered also as a linear functional on $g^{\mathbf{C}}$. By very clever means these spaces are shown to vanish for all j except $j =$ the negativity index $q(p, \Lambda)$ of the polarization (see 4.1). Moreover $H^{q(p, \Lambda)}(p \cap \operatorname{Ker} \Lambda, \pi^\infty_{-\Lambda})$ is one-dimensional; see Rosenberg's Theorem 2.4 in [74]; also see Penney [69]. With these remarks in mind an application of Theorem 4.3 gives

THEOREM 4.5 (J. Rosenberg–R. Penney 1979). *Let G be a connected, simply connected nilpotent Lie group and let p be a relatively ideal complex polarization at $\Lambda \in g^*$, $g =$ Lie algebra of G. Let $\pi^j(\Lambda, p, G)$ be the j-th harmonically induced representation defined in (4.2). Then $\pi^j(\Lambda, p, G)$ vanishes for $j \neq$ the negativity index $q(p, \Lambda)$ (see (4.1)). Moreover $\pi^{q(p, \Lambda)}(\Lambda, p, G)$ is irreducible and unitarily equivalent to the Kirillov representation π_Λ.*

Theorem 4.5 is clearly analogous to Theorem 3.11 and thus it represents the confirmation of a version of the Kostant-Langlands conjecture for nilpotent Lie groups. One may add that as a matter of fact the distinguished integer q_Λ in (3.9) is indeed the negativity index of a complex polarization-namely the polarization is a Borel subalgebra at a regular point.

5. FURTHER NOTES

1. We have pointed out earlier that in addition to Schmid's thesis work, early efforts towards proving the Kostant-Langlands conjecture were made by Narasimhan and Okamoto. The latter authors considered the special case when G/K admits a G invariant complex structure [1]). They constructed unitary representations $\pi_\Lambda^{0,j}$ of G on L_2-cohomology spaces associated to holomorphic vector bundles E_Λ over G/K induced by an irreducible unitary representation of K with highest weight Λ; compare remarks following (3.10). The $\pi_\Lambda^{0,j}$ are shown to be subject to an important *alternating sum formula* which, roughly stated, says that

(5.1) $$\sum_{j=0}^{n} (-1)^j \text{ character of } \pi_\Lambda^{0,j} = (-1)^{q_\Lambda} \text{ character of } \pi_\Lambda^*$$

[1]) Here G, K are as in section 3.

where π_Λ^* is the contragredient to the discrete class π_Λ given in Theorem 3.5, q_Λ is the number of non-compact positive roots α such that $(\Lambda+\delta, \alpha) > 0$ (compare (3.9)), and $n = \frac{1}{2} \dim_R G/K$. A precise statement of (5.1) is given in Theorem 1 of [60]. Once one has an alternating sum formula the class π_Λ^* can be realized by $\pi_\Lambda^{0, q_\Lambda}$ if a *vanishing theorem* $H_{\bar{\partial}, 2}^{0, j}(G/K, E_\Lambda) = 0$ for $j \neq q_\Lambda$ is proved. The methods of Narasimhan and Okamoto of establishing an alternating sum formula and a vanishing theorem served as a prototype for the work of later authors; see for example [59], [65], [99], [64], [98]. The vanishing theorems in [60] are improved by Parthasarathy in [64].

2. The ambitious program of decomposing a complex flag manifold under the action of a real group and of using the real group orbits as the setting for the geometric realization of unitary representations of semisimple (even reductive) Lie groups is carried out in the profound work of J. Wolf in [97], [99], [96]. A family of unitary representations which support the Plancherel measure are realized on *partially holomorphic* cohomology spaces. This family clearly contains many non-discrete classes. The realizations are similar to realizations by the Kostant-Kirillov method where one uses polarizations of semisimple orbits. However some interesting differences occur in the case when the reductive group has non-commutative Cartan subgroups; see [86], [98].

3. After (3.9) we remarked on the similarity in appearance of the Weyl and Harish-Chandra character formulas of Theorems 2.22 and 3.5. There is however a vast difference of roles which these formulas play. For example Weyl's formula determines the character *on all of the group* (since a compact group is covered by conjugates of a maximal torus) whereas conjugates of H in section 3 certainly do not cover G. It seems to be an extremely difficult problem (and perhaps an impossible one to solve) to obtain the character formula explicitely on an *arbitrary* Cartan subgroup H of a non-compact semisimple group.

4. A new proof of Harish-Chandra's regularity theorem for invariant eigendistributions (cf. remarks following (3.4) due to Atiyah and Schmid is now available; cf. [4], [7], [81]. [6] contains a new and largely self-contained account of the principal theory of the discrete series including existence theory, exhaustion, geometric realization, character formulae, and character behavior. These new methods rely on the Atiyah-Singer L_2-index theorem [3], [31], [32], [81], [83], [84].

5. It is possible to formulate Frobenius reciprocity for unitary representations on a Hilbert space $\mathscr{H}(D)$ of L_2-solutions of an invariant elliptic differential operator D on homogeneous bundles over a homogeneous space G/H whose isotropy subgroup H is compact modulo the center of G. Here G is a connected unimodular Lie group (not necessarily semisimple) subject to some mild structural constraints. In [33] Connes and Moscovici show that $\mathscr{H}(D)$ decomposes as a finite direct sum of irreducible unitary representations all of which are square-integrable modulo the center of G and occur with finite multiplicity. They derive for $\mathscr{H}(D)$ a reciprocity analogous to that expressed for the L_2-cohomology spaces in Theorem 3.15 and Theorem 4.3.

REFERENCES

[1] Ahiezer, D. Cohomology of compact homogeneous spaces. *Mat. Sb. 84* (126) (1971), 290-300 = *Math. USSR Sb. 13* (1971), 285-296.

[2] Andreotti, A. and E. Vessentini. Carleman estimates for the Laplace-Beltrami operator on complex manifolds. *Publ. I.H.E.S. No. 25* (1965), 313-362.

[3] Atiyah, M. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque 32-33* (1976), 43-72.

[4] —— *Characters of semisimple Lie groups.* Mimeographed lecture notes, Univ. of Oxford.

[5] —— The Harish-Chandra character. *Lecture No. 7 from Proceedings of the SRC/LMS Research Symposium on Representations of Lie Groups, London Math. Soc. Series 34*, Cambridge Univ. Press, 176-181.

[6] Atiyah, M. and W. Schmid. A geometric construction of the discrete series for semisimple Lie groups. *Inventiones Math. 42* (1977), 1-62.

[7] —— A new proof of the regularity theorem for invariant eigendistributions on semisimple Lie groups. *To appear.*

[8] Auslander, L. and B. Kostant. Polarization and unitary representations of solvable Lie groups. *Inventiones Math. 14* (1971), 255-354.

[9] Bernat, P., N. Conze, M. Duflo, M. Lévy-Nahas, M. Rais, P. Renouard et M. Vergne. *Représentations des groupes de Lie résolubles.* Monographies de la Société Mathématique de France, Dunod, éditeur. (1972) Paris.

[10] Blattner, R. On induced representations. *Amer. J. of Math. 83* (1961), 79-98.

[11] Borel, A. et A. Weil. Représentations linéaires et espaces homogènes Kählériens des groupes de Lie compacts. *Séminaire Bourbaki*, May 1954 (exposé by J.-P. Serre).

[12] Bott, R. Homogeneous vector bundles. *Annals of Math. (2) 66* (1957), 203-248.

[13] —— Induced representations. *Seminars on Analytic Functions, Vol. 2*, Institute for Advanced Study, Princeton, N.J., 1957.

[14] Bruhat, F. Travaux de Harish-Chandra. *Séminaire Bourbaki*, exposé 143 (1957), 1-9.

[15] Carmona, J. Représentations du groupe de Heisenberg dans les espaces de (0, q)-formes. *Math. Ann. 205* (1973), 89-112.

[16] Cartan, E. Les tenseurs irréductibles et les groupes simples et semisimples. *Bull. Sci. Math. 49* (1925), 130-152.

[17] CASSELMAN, W. and M. OSBORNE. The n-cohomology of representations with an infinitesimal character. *Compositio Math. 31* (1975), 219-227.

[18] HARISH-CHANDRA. On some applications of the universal enveloping algebra of a semisimple Lie algebra. *Trans. Amer. Soc. 70* (1951), 28-96.

[19] —— Representations of a semisimple Lie group on a Banach Space I. *Trans. Amer. Math. Soc. 75* (1953), 185-243.

[20] —— Representations of semisimple Lie groups II, III. *Trans. Amer. Math. Soc. 76* (1954), 26-65 and 234-253, respectively.

[21] —— Representations of semisimple Lie groups IV, V. *Amer. J. of Math. 77* (1955), 743-777 and 78 (1956), 1-41, respectively.

[22] —— Representations of semisimple Lie groups VI: Integrable and square-integrable representations. *Amer. J. of Math. 78* (1956), 564-628.

[23] —— The characters of semisimple Lie groups. *Trans. Amer. Math. Soc. 83* (1956), 98-163.

[24] —— Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc. 119* (1965), 457-508.

[25] —— Discrete series for semisimple Lie groups I. *Acta Math. 113* (1965), 241-318.

[26] —— Discrete series for semisimple Lie groups II: Explicit determination of the characters. *Acta Math. 116* (1966), 1-111.

[27] —— Two theorems on semisimple Lie groups. *Annals of Math. (2) 83* (1966), 74-128.

[28] —— Harmonic analysis on semisimple Lie groups. *Bull. Amer. Math. Soc. 76* (1970), 529-551.

[29] CHEVALLEY, C. *Theory of Lie groups I.* Princeton Math. Series, No. 8, Princeton Univ. Press, Princeton, N.J., 1946.

[30] CHEVALLEY, C. and S. EILENBERG. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Soc. 63* (1948), 85-124.

[31] CONNES, A. and H. MOSCOVICI. The L_2-index theorem for homogeneous spaces. *Bull. Amer. Math. Soc. 1* (1977), 688-690.

[32] —— The L_2-index theorem for homogeneous spaces of Lie groups. *To appear.*

[33] —— Invariant elliptic equations and discrete series representations. *Preprint I.H.E.S.*, March 1980.

[34] DEMAZURE, M. Une démonstration algébrique d'un théorème de Bott. *Inventiones Math. 5* (1968), 349-356.

[35] DOLBEAULT, P. Formes différentielles et cohomologie sur une variété analytique complexe I. *Annals of Math. (2) 64* (1956), 83-130.

[36] DUFLO, M. Représentations de carré intégrable des groupes semisimple réels. *Séminaire Bourbaki*, Vol. 1977/78, exposés 507-524, Springer-Verlag Lecture Notes No. 710, 22-40.

[37] FROBENIUS, G. Über relationen zwischen den characteren einer gruppe and ihrer untergruppen. *Sitzungsb Kön Preusz. Akad. Wiss. zu Berlin* (1898), 501-515.

[38] —— Über die composition der charactere einen gruppe. *Sitzungsb Kön Preusz, Akad. Wiss. zu Berlin* (1899), 330-339.

[39] GRIFFITHS, P. Some geometric and analytic properties of homogeneous complex manifolds Part I: Sheaves and cohomology. *Acta Math. 110* (1963), 115-155.

[40] GRIFFITHS, P. and W. SCHMID. Locally homogeneous complex manifolds. *Acta Math. 123* (1969), 253-302.

[41] GUNNING, R. and H. ROSSI. *Analytic functions of several complex variables.* Prentice-Hall, Englewood Cliffs, N.J., 1967.

[42] HELGASON, S. *Differential geometry and symmetric spaces.* Academic Press, New York and London, 1962.

[43] HIRZEBRUCH, F. *Topological methods in algebraic geometry.* Die Grundlehren der Math. Wissenschaften, Band 131, Springer-Verlag, Berlin, Heidelberg, New York, 1966.

[44] HOCHSCHILD, G. and J.-P. SERRE. Cohomology of Lie algebras. *Annals of Math. (2) 57* (1953), 591-603.

[45] HOTTA, R. On realization of the discrete series for semisimple Lie groups. *J. Math. Soc. Japan 23* (1971), 384-407.

[46] HUMPHREYS, J. *Introduction to Lie algebras and representation theory.* Graduate Texts in Math. Vol. 9, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[47] ISE, M. Some properties of complex analytic vector bundles over compact complex homogeneous spaces. *Osaka Math. J. 12* (1960), 217-252.

[48] JACOBSON, N. *Lie algebras.* Interscience Tracts in Pure and Appl. Math. No. 10, Interscience, New York, 1962.

[49] KIRILLOV, A. Unitary representations of nilpotent Lie groups. *Uspekhi Mat. Nauk. 17* (1962), 57-110 — Russian Math. Surveys 17 (1962), 53-104.

[50] KOSTANT, B. Lie algebra cohomology and the generalized Borel-Weil theorem. *Annals of Math. (2) 74* (1961), 329-387.

[51] —— Orbits, symplectic structures and representation theory. *Proc. U.S.-Japan Seminar Diff. Geom.*, Kyoto, Japan, 1965.

[52] KOSTANT, B. M.I.T. Lecture Notes *(unpublished)*, 1967.

[53] —— On certain representations which arise from a quantization theory. *Springer-Verlag Lecture Notes in Physics 6* (1970), 237-253, Berlin, Heidelberg, New York.

[54] LANGLANDS, R. Dimension of spaces of automorphic forms. *Proc. of Symposia in Pure Math. IX* (1966), 253-257.

[55] MACKEY, G. Induced representations of locally compact groups I. *Annals of Math. (2) 55* (1952), 101-139.

[56] —— Induced representations of locally compact groups II. *Annals of Math. (2) 58* (1953), 193-221.

[57] MOORE, C. and J. WOLF. Square integrable representations of nilpotent groups. *Trans. Amer. Math. Soc. 185* (1973), 445-462.

[58] MOSCOVICI, H. A vanishing theorem for L^2-cohomology in the nilpotent case. *Non-commutative harmonic analysis*, Springer-Verlag Lecture Notes in Math. 728 (1979), 201-210, Berlin, Heidelberg, New York.

[59] MOSCOVICI, H. and A. VERONA. Harmonically induced representations of nilpotent Lie groups. *Inventiones Math. 48* (1978), 61-73.

[60] NARASIMHAN, M. and K. OKAMOTO. An analogue of the Borel-Weil-Bott theorem for hermitian symmetric pairs of non-compact type. *Annals of Math. (3) 91* (1970), 486-511.

[61] OKAMOTO, K. On induced representations. *Osaka J. of Math. 4* (1967), 85-94.

[62] —— On square-integrable $\bar{\partial}$-cohomology spaces attached to homogeneous symplectic manifolds. Unpublished manuscript, Institute for Advanced Study, Princeton, N.J.

[63] OKAMOTO, K. and H. OZEKI. On square-integrable $\bar{\partial}$-cohomology spaces attached to hermitian symmetric spaces. *Osaka J. of Math. 4* (1967), 95-110.

[64] PARTHASARATHY, R. A note on the vanishing of certain L_2-cohomologies. *J. of Math. Soc. Japan 23* (1971), 676-691.

[65] —— Dirac operator and the discrete series. *Annals of Math. 96* (1972), 1-30.

[66] —— An algebraic construction of a class of representations of a semisimple Lie algebra. *Math. Ann. 226* (1977), 1-52.

[67] PENNEY, R. Canonical objects in the Kirillov theory of nilpotent Lie groups. *Proc. Amer. Math. Soc. 66* (1977), 175-178.

[68] —— Harmonically induced representations of nilpotent Lie groups and automorphic forms on nilmanifolds. *Trans. Amer. Math. Soc. 260* (1980), 123-145.

[69] —— Lie cohomology of representations of nilpotent Lie groups and holomorphically induced representations. *Preprint, 1979.*

[70] PETER, F. und H. WEYL. Die vollständigkeit der primitiven darstellungen einer geschlossenen kontinueuirlichen gruppe. *Math. Ann. 97* (1927), 727-755.

[71] *Proceedings of Symposia in Pure Math., Harmonic analysis on homogeneous spaces,* vol. 26 (1973), A.M.S., Providence, R.I.

[72] PUKANSZKY, L. *Leçons sur les représentations des groupes.* Monographies de la Société Mathématique de France, Dunod, éditeur (1967), Paris.

[73] RAMANAN, S. Holomorphic vector bundles on homogeneous spaces. *Topology 5* (1966), 159-177.

[74] ROSENBERG, J. Realization of square-integrable representations of unimodular Lie groups on L^2-cohomology spaces. *Trans. Amer. Math. Soc. 261* (1980), 1-32.

[75] SATAKE, I. Unitary representations of a semi-direct product of Lie groups on $\bar{\partial}$-cohomology spaces. *Math. Ann. 190* (1971), 177-202.

[76] SCHIFFMAN, G. Un analogue du théorème de Borel-Weil-Bott dans le cas non compact. *Séminaire Bourbaki,* June 1971, 323-336.

[77] SCHMID, W. *Homogeneous complex manifolds and representations of semisimple Lie groups.* Ph.D. thesis, Univ. Calif., Berkeley, 1967.

[78] —— On a conjecture on Langlands. *Annals of Math. 93* (1971), 1-42.

[79] —— Some properties of square-integrable representations of semisimple Lie groups. *Annals of Math. 102* (1975), 535-564.

[80] —— On the characters of the discrete series. *Inventiones 30* (1975), 47-144.

[81] —— Representations of semisimple Lie groups. *Lecture No. 8 from Proceedings of the SRC/LMS Research Symposium on Representations of Lie Groups, London Math. Soc. Series 34,* Cambridge Univ. Press, 185-235.

[82] —— L^2-cohomology and the discrete series. *Annals of Math. 103* (1976), 375-394.

[83] —— A geometric construction of the discrete series for semisimple Lie groups. *Lectures from the Nato Advanced Study Institute on Harmonic Analysis and Representations of Semisimple Lie Groups held at Liege,* Belgium, Sept. 1977, edited by M. Cahen, M. DeWilde, and J. Wolf; D. Reidel Publ. Company.

[84] SINGER, I. Some remarks on operator theory and index theory, from K-theory and Operator Algebras. *Springer-Verlag Lecture Notes in Math. 575* (1975), 128-138, Berlin, Heidelberg, New York.

[85] VARADARAJAN, V. *Harmonic analysis on real reductive groups.* Springer-Verlag Lecture Notes in Math. 576 (1977), Berlin, Heidelberg, New York.

[86] WAKIMOTO, M. Polarizations of certain homogeneous spaces and most continuous principal series. *Hiroshima Math. J. 2* (1972), 483-533.

[87] WALLACH, N. Induced representations of Lie algebras and a theorem of Borel-Weil. *Trans. Amer. Math. Soc. 136* (1969), 181-187.

[88] —— *Harmonic analysis on homogeneous spaces.* Marcel Dekker, Inc., New York, 1973.

[89] —— Representations of semisimple Lie groups and Lie algebras. *Proceedings of the 1977 Seminar of the Canadian Math. Congress on Lie Theories and Their Applications, Queen's Papers in Pure and Applied Math. No. 48,* 154-246.

[90] WANG, H. Closed manifolds with homogeneous complex structure. *Amer. J. of Math. 76* (1954), 1-32.

[91] Warner, G. *Harmonic analysis on semisimple Lie groups, I, II.* 1972, Springer-Verlag, Vol. 188, 189, Berlin, Heidelberg, New York.

[92] Wells, R., Jr. *Differential analysis on complex manifolds.* Prentice-Hall, Englewood Cliffs, N.J., 1973.

[93] Weyl, H. Theorie der darstellung kontinuerlichen halb-einfacher gruppen durch lineare transformationen. Teil I, *Mathematische Zeitschrift 23* (1925), 271-309, Teil II, *24* (1926), 328-376, and Teil III, *24* (1926), 377-395.

[94] —— *The classical groups.* Princeton Univ. Press, Princeton, N.J., 1946.

[95] Williams, F. Complex homogeneous bundles and finite-dimensional representation theory. *Proceedings of Symposia in Pure Math., Several complex variables, Vol. 30, Part 2* (1977), 317-320.

[96] Wolf, J. The action of a real semisimple group on a complex flag manifold, I : Orbit structure and holomorphic arc components. *Bull. Amer. Math. Soc. 75* (1969), 1121-1237.

[97] —— Complex manifolds and unitary representations. *Proceedings of the International Conference in Several Complex Variables*, Univ. Maryland, 1970, Springer-Verlag Lecture Notes in Math. 185 (1971), 242-287, Berlin, Heidelberg.

[98] —— Orbit method and nondegenerate series. *Hiroshima Math. J. 4* (1974), 619-628.

[99] —— The action of a real semisimple group on a complex flag manifold, II : Unitary representations on partially holomorphic cohomology spaces. *Memoirs Amer. Math. Soc., No. 138*, 1974.

[100] Zelebenko, D. *Compact Lie groups and their representations.* « Nauka », Moscow, 1970 ; English transl. Transl. Math. Monographs, Vol. 40, Amer. Math. Soc., Providence, R.I., 1973.

(Reçu le 20 janvier 1981)

Floyd L. Williams

University of Massachusetts
Amherst, MA 01003
USA