1. Maneuvering a needle

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SOME PARADOXICAL SETS WITH APPLICATIONS IN THE GEOMETRIC THEORY OF REAL VARIABLE [1]

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The purpose of this paper is to present a small excursion through a certain area of the theory of real variable, describing some strange constructions, paradoxical and beautiful in their own way, that have recently come to illuminate some other important topics in related fields such as Fourier analysis. We shall do it in an expository way, trying to avoid most of the technicalities. For them we refer the reader to the works of the author published in 1975 and 1981.

1. MANEUVERING A NEEDLE

In 1917 Kakeya proposed a curious problem with the aspect of a puzzle. It can be formulated in the following way. Let us consider a one-dimensional car, like a straight needle of one meter of length, located on the plane. We can maneuver the needle on its plane in a continuous way until placing it in the same plane it occupies but in inverted position. In doing so the needle will sweep a certain area. The question is: *What is the minimal value of the areas of the figures within which the needle can be continuously inverted?*

A circle of radius 1/2 (area: $\pi/4 = 0.785398...$) with center at the middle point of the needle is such a figure in which the needle can rotate (Fig. 1).

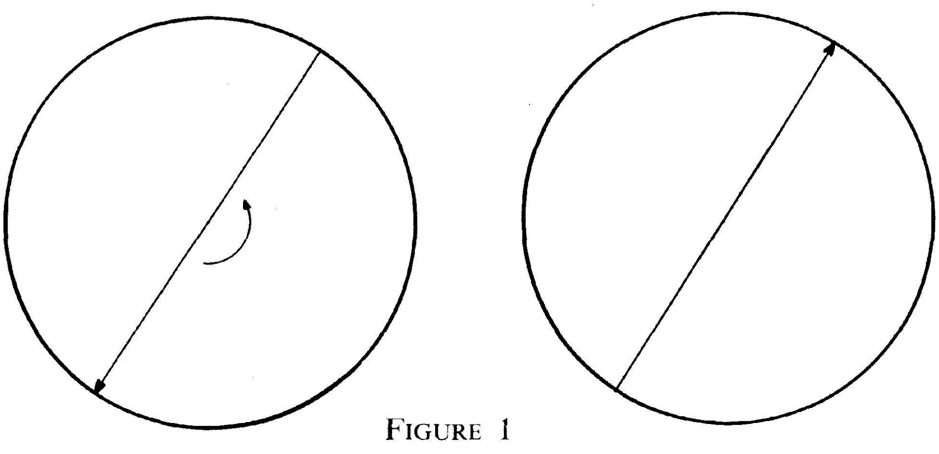

FIGURE 1

[1]) The present paper is an expanded version of a talk given at the Mathematics Department of the Universidade Federal do Rio de Janeiro.

But also the triangle of Figure 2 (area: $1/\sqrt{3} = 0.5773502... < \pi/4$) is another such figure of smaller area.

For a long time many people thought that the problem would be solved by means of a sort of curvilinear triangle (see Fig. 3), a figure bounded by a hypocycloid γ with three cuspidal points inscribed in a circle of radius 3/4. This curve has the property that for each point M in γ the tangent at M to γ intersects γ in two other points A and B such that the length of AB is 1. The area of the figure enclosed by it is $\pi/8 = 0.392699...$ and it is easy to see that a needle of length 1 can turn around in any open set containing such a figure inside.

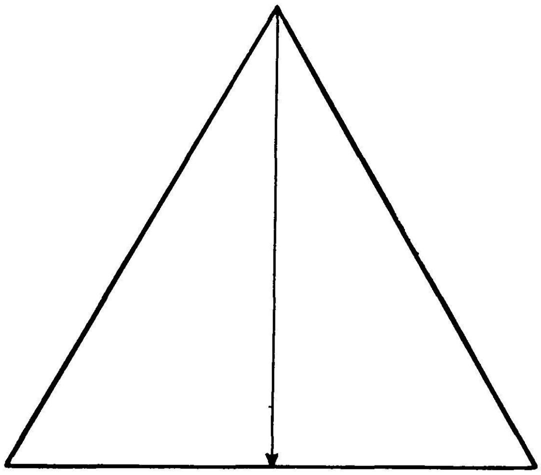

FIGURE 2

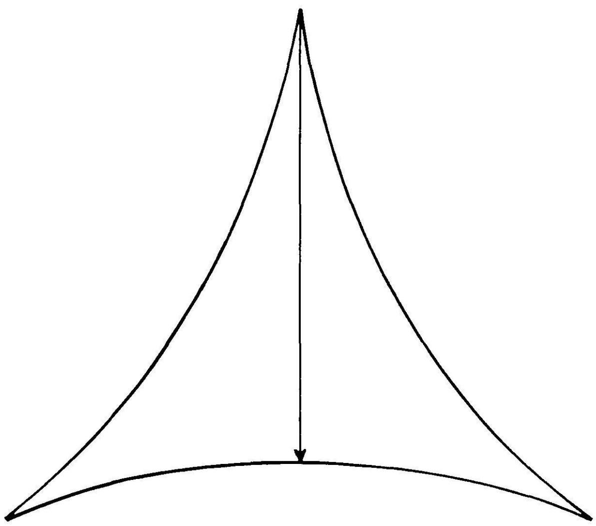

FIGURE 3

2. STREETS IN ALL DIRECTIONS COVERING NULL AREA

Kakeya's problem was in fact solved in 1919 by Besicovitch, but nobody, not even Besicovitch himself, realized it. He was at that time in Perm, in a University rather isolated from the rest of the mathematical world where the Kakeya problem did not arrive. On the other hand the tool created by Besicovitch for the solution of some other problem did not go very far from his place either. Besicovitch constructed a plane set of null area containing segments of length 1 in all directions. As we shall see this gives the following solution to the Kakeya problem: *Given any arbitrarily small* $\eta > 0$ *one can construct a plane figure with area smaller than* η *such that the needle can be continuously inverted inside it.*