THE CLEBSCH-GORDAN FORMULAS

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by Daniel FLATH

0. INTRODUCTION

The explicit decomposition of tensor products of irreducible representations is of fundamental importance in many applications of representation theory. For finite dimensional representations of the Lie algebra $\mathfrak{sl}_2$ definitive results are contained in the famous Clebsch-Gordan formulas which are constantly and routinely used by physicists in applying the quantum theory of angular momentum. We give in this article a presentation and derivation of equivalent results, Theorems 5.1 and 5.4.

We shall base a study of the representations Hom(V, W) (rather than $V \otimes W$) for irreducible $\mathfrak{sl}_2$-representations V and W on the analysis of a Weyl algebra $\mathscr{A}$ of polynomial differential operators in two variables. This point of view is one developed in a recent attack on the Clebsch-Gordan problem for $\mathfrak{sl}_3$ [2].

The usefulness of the Weyl algebra in the resolution of the Clebsch-Gordan problem is well-known. For years physicists have worked with it under the name "boson calculus" [1]. One mathematical reference is [3]. Nothing in the present article is new except possibly the arrangement of the proofs which has been made with the benefit of experience gained working with $\mathfrak{sl}_3$. It seems to me that this arrangement has a naturalness and simplicity to recommend it.

I would like to thank L. C. Biedenharn for interesting discussions on the subject of this paper.

1. SOME REPRESENTATIONS OF $\mathfrak{sl}_2$

Let $V = \mathbf{C}[X, Y]$, the vector space of polynomials in two variables X and Y. For integers m let V_m be the subspace of homogeneous polynomials of degree m, with $V_m = (0)$ for negative m.

Let $SL_2(\mathbf{C})$ act linearly on V as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \tag{1.1}$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \qquad \text{for } g \in SL_2(\mathbf{C}). \tag{1.2}$$

Each V_m is an $SL_2(\mathbf{C})$ subrepresentation of V.

By $\mathfrak{sl}_2$ we denote the Lie algebra of 2×2 complex matrices with trace 0. The representation of $SL_2(\mathbf{C})$ on V gives rise, through differentiation, to a representation of $\mathfrak{sl}_2$ on V.

$$L \cdot v = \frac{d}{dt}\bigg|_{t=0} \exp(tL) \cdot v \qquad \text{for } L \in \mathfrak{sl}_2, v \in V. \tag{1.3}$$

Choose a basis E_+, E_-, H of $\mathfrak{sl}_2$ as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

An easy calculation establishes the following equalities of linear endomorphisms of V.

$$E_+ = X\partial_Y, \qquad E_- = Y\partial_X, \tag{1.5}$$

$$H = X\partial_X - Y\partial_Y. \tag{1.6}$$

From (1.5) and (1.6) one easily deduces that each V_m is an *irreducible* representation of $\mathfrak{sl}_2$ (and of $SL_2(\mathbf{C})$).

We define for integers m, n a representation τ of $\mathfrak{sl}_2$ on $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$ by means of formula (1.7).

$$(\tau(L) \cdot T)v = L(Tv) - T(Lv)$$
$$\text{for } L \in \mathfrak{sl}_2, T \in \mathrm{Hom}_{\mathbf{C}}(V_m, V_n), v \in V_m. \tag{1.7}$$

The principal result of this article is the explicit decomposition of the $\mathfrak{sl}_2$-representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

2. The Weyl algebra $\mathscr{A}$

Let $\mathscr{A}$ be the subalgebra of $\mathrm{End}_{\mathbf{C}}(V)$ consisting of polynomial differential operators on $V = \mathbf{C}[X, Y]$. The algebra $\mathscr{A}$ is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X{}^a \partial_Y{}^b. \tag{2.1}$$

The Euler operator J, which acts as scalar multiplication by m on V_m, lies in $\mathscr{A}$.

$$J = X\partial_X + Y\partial_Y . \tag{2.2}$$

The next lemma assures us that $\mathscr{A}$ is large enough for the study of all spaces $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

LEMMA 2.3. *Let U be a finite dimensional vector subspace of V and let $T \in \mathrm{End}_{\mathbf{C}}(U)$. Then there exists an element of $\mathscr{A}$ whose restriction to U equals T.*

Proof: The element $S = X^c Y^d (\partial_X)^a (\partial_Y)^b \prod_{\substack{m=0 \\ m \neq a+b}}^{N} (J-m)$ of $\mathscr{A}$ maps $X^a Y^b$ to a nonzero multiple of $X^c Y^d$ and kills all other monomials of degree at most N. But by enlarging U we may assume that $\mathrm{End}_{\mathbf{C}}(U)$ is spanned by restrictions of elements of the form S. □

We use the inclusion of $\mathfrak{sl}_2$ in $\mathscr{A}$ to define a representation ρ of $\mathfrak{sl}_2$ on $\mathscr{A}$.

$$\rho(L)a = [L, a] \qquad \text{for } L \in \mathfrak{sl}_2, a \in \mathscr{A} . \tag{2.4}$$

For integers n let $\mathscr{A}^n$ be the set of T in $\mathscr{A}$ such that $T(V_m) \subset V_{m+n}$ for all m.

This defines a grading of $\mathscr{A}$ which is preserved by the action of $\mathfrak{sl}_2$.

$$\mathscr{A} = \bigoplus_{n \in \mathbf{Z}} \mathscr{A}^n , \qquad \mathscr{A}^m \cdot \mathscr{A}^n \subset \mathscr{A}^{m+n} , \tag{2.5}$$

$$\rho(L)\mathscr{A}^n \subset \mathscr{A}^n \qquad \text{for all } L \in \mathfrak{sl}_2 . \tag{2.6}$$

The algebra $\mathscr{A}$ and representation ρ have been defined just so that the next lemma, which is an immediate consequence of Lemma 2.3, will be true.

LEMMA 2.7. *For each m, n the restriction map*

$$\mathrm{res}: \mathscr{A}^n \to \mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$$

is a surjective homomorphism of $\mathfrak{sl}_2$ representations. □

The method of this paper is to deduce the decomposition of the representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ from the decomposition of the representation ρ on $\mathscr{A}$ by means of Lemma 2.7.

3. The theory of the highest weight

Before decomposing the $\mathfrak{sl}_2$-space $\mathscr{A}$ we must review the finite dimensional representation theory of $\mathfrak{sl}_2$.

The *weight vectors* of an $\mathfrak{sl}_2$-representation W are the eigenvectors of H in W. The *weights* of W are the eigenvalues of its nonzero weight vectors.

Every finite dimensional $\mathfrak{sl}_2$-module is spanned by its weight vectors. The weights of such a representation are all integers and are thus ordered by the usual order on **R**. The largest of a finite set of integral weights is traditionally referred to as the *highest* weight.

Two finite dimensional irreducible $\mathfrak{sl}_2$-representations are isomorphic if and only if they have the same highest weights, which are necessarily nonnegative.

The element $X^a Y^b$ of V is a weight vector of weight a-b. This shows that X^m is a vector of highest weight m in V_m and therefore that the V_m for $m \geqslant 0$ form a set of representatives of the equivalence classes of finite dimensional irreducible $\mathfrak{sl}_2$-representations; which is precisely why we are studying them is this paper.

The last general fact which we will recall without proof is this: every finite dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducible representations.

Given a representation W of $\mathfrak{sl}_2$ which is a sum of finite dimensional representations one often wishes to write it explicitly as a direct sum of irreducible representations, that is, of representations isomorphic to the V_m. A method for doing this is provided by the observation that the space of weight vectors of highest weight in V_m is the space annihilated by E_+ and is one dimensional. Thus for each $v \in W$ of weight m such that $E_+ v = 0$, there is a unique $\mathfrak{sl}_2$-homomorphism from V_m to W taking X^m to v. The explicit decomposition of W therefore amounts to the determination of a basis consisting of weight vectors of the kernel of E_+ in W.

4. The decomposition of $\mathscr{A}$

We apply the procedure of the last paragraph to the representation of $\mathfrak{sl}_2$ on $\mathscr{A}$. By definition of ρ the kernel of $\rho(E_+)$ is just the commutant of E_+ in $\mathscr{A}$.

Let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ generated by X, ∂_Y, and J.

PROPOSITION 4.1. *$\mathscr{B}$ is the commutant of E_+ in $\mathscr{A}$.*

Proof: One easily verifies that E_+ commutes with X, ∂_Y, and J, which shows that $\mathscr{B}$ is contained in the commutant of E_+.

Let U be the $\mathfrak{sl}_2$-subrepresentation of $\mathscr{A}$ generated by $\mathscr{B}$. The considerations of Section 3 show that the inclusion of the commutant of E_+ in $\mathscr{B}$ is equivalent to the assertion that U equals all of $\mathscr{A}$. We proceed to establish that equality.

The algebra $\mathscr{B}$ is spanned as a vector space by the elements

$$J^a X^b (\partial_Y)^c \qquad \text{with } a, b, c \geqslant 0 . \tag{4.2}$$

We present two calculations.

$$[E_-, J^a X^b (\partial_Y)^{c+1}] = -(b+c+1) J^a X^b (\partial_Y)^c \partial_X + b(J+1-b) J^a X^{b-1} (\partial_Y)^c \tag{4.3}$$

$$[E_-, J^a X^{b+1} (\partial_Y)^c] = (b+1) J^a X^b (\partial_Y)^c Y - c J^a X^b (\partial_Y)^{c-1} (b+1+X\partial_X) . \tag{4.4}$$

From (4.3) one concludes that $\mathscr{B} \cdot \partial_X \subset U$. From that and (4.4) one concludes that $\mathscr{B} \cdot Y \subset U$.

Because E_- commutes with ∂_X and Y, one has that

$$\rho(E_-)^n (\mathscr{B} \partial_X) = \big(\rho(E_-)^n \mathscr{B}\big) \cdot \partial_X$$

and that

$$\rho(E_-)^n (\mathscr{B} \cdot Y) = \big(\rho(E_-)^n \mathscr{B}\big) \cdot Y .$$

Because $V_m = \bigoplus_{n=0}^{\infty} E_-{}^n (\mathbf{C} X^m)$ one knows that $U = \bigoplus_{n=0}^{\infty} \rho(E_-)^n \mathscr{B}$. And thus

$$U \cdot \partial_X \subset U , \qquad U \cdot Y \subset U . \tag{4.5}$$

Iterating, we have

$$U Y^d (\partial_X)^e \subset U \qquad \text{for } d, e, \geqslant 0 . \tag{4.6}$$

But $\mathscr{A}$ is generated as an algebra by X, Y, ∂_X, and ∂_Y and so (4.2) and (4.6) prove that $U = \mathscr{A}$. □

COROLLARY 4.7. *$\mathscr{A}^0$ is the subalgebra of $\mathscr{A}$ generated by $\mathfrak{sl}_2$ and J.*

Proof: $\mathscr{A}^0$ is the $\mathfrak{sl}_2$-subrepresentation of $\mathscr{A}$ generated by $\mathscr{A}^0 \cap \mathscr{B}$.

$\mathscr{A}^0 \cap \mathscr{B}$ is spanned by the elements (4.2) such that $b = c$, all of which are of the form $J^a E_+{}^b$. □

We remark that the subalgebra of $\mathscr{A}$ generated by $\mathfrak{sl}_2$ is canonically isomorphic to the universal enveloping algebra of $\mathfrak{sl}_2$. The element $J(J+2)$ equals $H^2 + 2(E_+E_- + E_-E_+)$, the Casimir element for $\mathfrak{sl}_2$. Thus $\mathscr{A}^0$ is a little larger than the enveloping algebra of $\mathfrak{sl}_2$.

For integers l, n define $\mathscr{B}\begin{pmatrix} n \\ l \end{pmatrix}$ to be the set of $T \in \mathscr{B} \cap \mathscr{A}^n$ such that $\rho(H)T = lT$.

This defines a grading of $\mathscr{B}$:

$$\mathscr{B} = \oplus \mathscr{B}\begin{pmatrix} n \\ l \end{pmatrix}, \qquad \mathscr{B}\begin{pmatrix} n \\ l \end{pmatrix} \cdot \mathscr{B}\begin{pmatrix} n' \\ l' \end{pmatrix} \subset \mathscr{B}\begin{pmatrix} n+n' \\ l+l' \end{pmatrix}. \tag{4.8}$$

The generators of $\mathscr{B}$ fit in as follows:

$$J \in \mathscr{B}\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad X \in \mathscr{B}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \partial_Y \in \mathscr{B}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{4.9}$$

PROPOSITION 4.10. *i)* $\mathscr{B}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{C}[J]$.

ii) $\mathscr{B}\begin{pmatrix} n \\ l \end{pmatrix} \neq 0$ *if and only if* $l \geqslant 0, |n| \leqslant l,$ *and* $l \equiv n \pmod 2$. *If these conditions are met, then*

$$\mathscr{B}\begin{pmatrix} n \\ l \end{pmatrix} = \mathbf{C}[J] \cdot X^{\frac{l+n}{2}} (\partial_Y)^{\frac{l-n}{2}} \tag{4.11}$$

Proof: Immediate. □

We note that the condition that $\mathscr{B}\begin{pmatrix} n \\ l \end{pmatrix} \neq (0)$ may be rephrased thus: $l \geqslant 0$ and n is a weight of V_l.

5. DECOMPOSITION OF $\mathrm{Hom}(V_m, V_{m+n})$

THEOREM 5.1. *Let* l, m, n *be integers with* $l, m, m+n \geqslant 0$. *There is an* $\mathfrak{sl}_2$*-subrepresentation of* $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *which is isomorphic to* V_l *if and only if* $|n| \leqslant l, n \equiv l \pmod 2$, *and* $m \geqslant \dfrac{l-n}{2}$.

Moreover, when these conditions are met there is a unique such subrepresentation. A weight vector of weight l *in it is given by*

$$X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}.$$

Proof: By Lemma 2.7 and the definition of $\mathscr{B}\binom{n}{l}$, a weight vector of weight l of the subrepresentation sought must be the restriction to V_m of an element of $\mathscr{B}\binom{n}{l}$. By Lemma 4.10ii, all such restrictions are scalar multiples of the restriction of $X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}$ to V_m, which restriction is nonzero only when $m \geqslant \frac{l-n}{2}$. □

It is interesting to observe that the weight l weight vector in $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ given by Theorem 5.1 is "independent" of m.

Finally we want to give formulas for the weight vectors in $\mathrm{Hom}(V_m, V_{m+n})$ of all weights, not just of highest weight.

For integers l, i, j with $l \geqslant 0$ and $0 \leqslant i,j \leqslant l$, define an element $A_l(i,j)$ of $\mathscr{A}$:

$$A_l(i,j) = \sum_{\alpha \leqslant k \leqslant \beta} (-1)^k \binom{l}{i}\binom{i}{k}\binom{l-i}{j-k} X^{l-i-j+k}\, Y^{j-k}(\partial_X)^k(\partial_Y)^{i-k}$$

with $\alpha = \sup\{0, i+j-l\}$ and $\beta = \inf\{i,j\}$. (5.2)

LEMMA 5.3. $\rho(E_-)^j \binom{l}{i} X^{l-i}(\partial_Y)^i = j!\, A_l(i,j)$.

Proof: By induction on j. Use the formula:

$$[E_-, D(i,j,a,b)] = iD(i-1, j+1, a, b) - bD(i,j,a+1,b-1)$$

with D as in (2.1). □

THEOREM 5.4. *Let* l, m, n *be such that there is a subrepresentation of* $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *isomorphic to* V_l. *Then an inclusion of representations* $\phi: V_l \to \mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *may be given by the formula:*

$$\phi(X^{l-j}Y^j) = \frac{1}{\binom{l}{j}} A_l\left(\frac{l-n}{2}, j\right). \tag{5.5}$$

Proof: This depends on (5.3) and the calculation in V_l that

$$E_-{}^jX^l = \frac{l!}{(l-j)!}X^{l-j}Y^j. \qquad \square$$

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