# Affine structures in 2, 3, and 4 dimensions

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space $M'$ is obtained by gluing, each time along one of the two segments of $a$ or $c$, as many copies of open sectors bounded by the lines $a$ and $c$, (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the $t = 1$ flow map. In suitable homogeneous coordinates the last is expressed as $f_1 : f_t : (x, y, z) \to (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t})\ \alpha < \beta < \gamma, \quad t = 1.$

*Remark.* Following the curve from its initial point $P$ to its endpoint $P'$, one can say that the sectors of $P$ and $P'$ were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \to (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with $f_1$:

$$g : (x, y, z) \to (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

$\lambda, \mu, \nu \in \mathbf{R}$.

## Affine structures in 2, 3, and 4 dimensions

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$ (where $\mathbf{R}^* = \mathbf{R} - \{0\}$) lifts to an affine transformation of $V = \mathbf{R}^3 - \{0\}$, unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number $\alpha > 1$ (e.g. $\alpha = 2$).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in $V$ etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops over the part outside the coordinate axes of $\{X > 0\} \cup \{Z > 0\} \subset \mathbf{R}^3 = \{(x, y, z)\}$, but not as a covering. In these examples the 3-manifold $M$ is a 3-torus.

ii) Similarly, a projective transformation of the complex projective line $\mathbf{CP}^1 = \mathbf{C}^2 - \{0\}/\mathbf{C}^*$, that is to say an orientable conformal or inversive transformation of $S^2 = \mathbf{CP}^1$, lifts to a complex affine transformation of $V = \mathbf{C}^2 - \{0\}$, unique but for scalar multiplication and commuting with scalar multiplication.

We can build a four dimensional affine manifold from an inversive 2-manifold, which is actually a complex affine manifold of **C**-dimension 2, and this construction is the analogue of the above over **C**, thinking of $S^2$ as $\mathbf{CP}^1$ and the conformal transformations as the **C**-projective transformation.

Again compactness is achieved if we divide by $\alpha = 2$. Thus using the inversive Example 2 we obtain affine 4-manifolds whose developing image has a complicated boundary related to the non-differentiable Jordan curve. Using Example 3, we obtain an affine four-manifold whose developing image in $R^4$ omits a Cantor set of two planes passing through the origin.

Using Example 4, we can build affine manifolds whose developing map is not a covering of its image (which is all of $\mathbf{C}^2 - 0$). And we repeat, all the above are actually complex affine structures on compact 4-manifolds.

NOTE 1 (see page 16). *Ehresmann* defined the *development map* as follows. Let $\mathscr{P} \to M$ be the principle $\mathscr{A}$-bundle over $M$, whose points are germs $[x, \kappa]$ of canonical charts $\{x \in U \subset M, \kappa\colon U \to A\}$. Define a new topology $\mathscr{F}(\mathscr{P})$ in the set $\mathscr{P}$ by taking as open set the germs at all points $x \in U$ of any given chart $\kappa\colon U \to A$. The natural map $d\colon \mathscr{F}(\mathscr{P}) \to A$ is an immersion. Choose one component of $\mathscr{F}(\mathscr{P})$ and call it $M'$. The restriction $d\colon M' \to A$ is a development map. The restriction of the natural fibre bundle projection $p\colon \mathscr{F}(\mathscr{P}) \to M$ is a covering $M' \to M$.

NOTE 2 (see page 16). *The fibre bundle picture.* For the simple *local* discussion of *one canonical chart* $U \subset A$, we can describe a trivial fibre bundle $E_U = U \times A \to U$ by assigning to any $x \in U$ the "heavily osculating" model space $A_x = A$. The manifold $U$ is embedded as the diagonal cross section. $s(U) = \{(y, y)\} = \text{diag}(U \times U) \subset U \times U \subset U \times A$. Its points are the points of tangency of fibre and base manifolds. Finally a foliation $\mathscr{F}$ is defined as the one with horizontal leaves $U \times \{v\} \subset E_U = U \times A$, for $v \in A$.

For the *global* discussion of an $\mathscr{A}$-structure on a manifold $M$, we assume $\mathscr{A}$-compatible canonical charts that are topological embeddings $\kappa: U \hookrightarrow A$ for small open sets $U \subset M$. A point of the fibre bundle space $E$ over $M$ is by definition a triple

$$\{x, \kappa, v\},$$

where $x \in U \subset M, \kappa : U \to A$ is a canonical chart and $v \in A$, *modulo equivalence* by the action of $\mathscr{A}$ given by $g : \{x, \kappa, v\} \rightrightarrows \{x, \kappa', v'\}$ where $\kappa' = g \circ \kappa$, $v' = gv$, $g \in \mathscr{A}$. In $E$, $M$ is embedded as the "diagonal cross section" $s(M)$, whose points are represented by triples $\{x, \kappa, \kappa(x)\}$. The foliation $\mathscr{F}$ has the local "horizontal" leaves represented by triples $\{U, \kappa, v\}$. For contractible closed curves starting and ending at $x_0 \in M$ in the base space $M$, the holonomy of the foliation is of course the identity map of the fibre $A_{x_0}$. As a consequence for closed curves in general, starting and ending at $x_0$ the holonomy gives the representation of $\pi_1 M$ into the group $\mathscr{A}$ acting on $A_{x_0}$. "Parallel displacement" of the points of $s(M)$ along the lifting in $\mathscr{F}$-leaves of curves in the base space ending at $x_0$, determines the development map $M' \to A_{x_0}$.

NOTE 3 (see page 16). *Flat Cartan connections.* Manifolds with canonical $(\mathscr{A}, A)$-charts are the flat cases (without torsion and without curvature) of *manifolds $M$ with a general $(\mathscr{A}, A)$-connection.* They are defined in [4] as follows

(1) A fibre bundle $A \to E \to M$ with fibre $A$ over $M$

(2) A fixed cross section $s(M)$

(3) An $n$-plane field $\xi$ in $E$ transversal to the fibres and transversal to the fixed cross sections, such that

(4) The holonomy obtained by lifting a closed curve starting and ending at $x_0 \in M$, into all curves tangent to $\xi$, belongs to $\mathscr{A}$ acting on $A_{x_0}$. It is in general different for homotopic curves. It is flat if contractible closed curves have trivial holonomy (= identity).

The development of a curve ending in $x_0$ in $M$, is obtained by dragging along $\xi$ the corresponding points of $s(M)$ until they arrive in the fibre $A_{x_0}$. In the flat case homotopic curves with common initial and end points give the same image of the initial point in the end fibre and the development map is achieved.