§5. Global knot theory in brief—the projective case

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Finally, some conjectures on less algebraic knot invariants of links of singularities should be mentioned. The *Milnor number* μ of a singularity is the rank of $H_1(F_0; \mathbf{Z})$. Let us look at a single branch, for convenience. Then Milnor conjectured [Mi 2] that $\mu/2$, which is the genus of F_0 and therefore (by a general theorem about fibred links) the genus of the knot ∂F_0, actually is the *slice genus* of ∂F_0. One can make the weaker conjecture that at least $\mu/2$ is the *ribbon genus* of ∂F_0. Milnor also wondered if this integer equalled the *Überschneidungszahl*, or *gordian number*, of ∂F_0; again the conjecture can be weakened, if one introduces the concepts of "slice Überschneidungszahl" and "ribbon Überschneidungszahl", cf. [Ru 2]. The conjectures are true in various cases where direct calculations can be made (e.g., the cusps $z = t^2$, $w = t^3$), but I know of no general results.

§5. Global knot theory in brief—the projective case

A curve $\Gamma \subset \mathbf{CP}^2$ can be given by its resolution $r: R \to \Gamma$ (a complex-analytic map from a compact Riemann surface into $\mathbf{CP}^2$ which is generically one-to-one on R) or by its polynomial $F(z_0, z_1, z_2) \in \mathbf{C}[z_0, z_1, z_2]$ (the homogeneous polynomial of least degree, not identically zero, which vanishes at every point of Γ). These suggest different kinds of knot-theoretical questions. One can consider all curves with diffeomorphic resolutions (the requirement that the curves have complex-analytically equivalent resolutions would be too stringent, and is less topological), and ask how differently they can be placed in the plane. Or one can consider families of curves, each cut out by a polynomial of some fixed degree.

Let P_d denote the projective space of the vector space of homogeneous complex polynomials in (z_0, z_1, z_2) of degree d. Because we never want to consider curves with multiple components, we throw out of P_d the algebraic subset corresponding to reducible polynomials with a multiple factor; the remaining Zariski-open subset Q_d corresponds to the set of what we may call curves of *geometric degree* d. If (the equivalence class of) $F(z_0, z_1, z_2)$ belongs to P_d, let $\Gamma_F = \{(z_0 : z_1 : z_2) \in \mathbf{CP}^2 : F(z_0, z_1, z_2) = 0\}$; then $F \in Q_d$ if and only if there is an open dense set of lines in $\mathbf{CP}^2$ which intersect Γ_F transversely in d distinct points.

The condition that Γ_F have a singular point is, of course, an algebraic condition on F. Let $S_d \subset P_d$ be the algebraic subset of singular curves without multiple components, and $R_d = Q_d - S_d$ the Zariski-open subset of polynomials of *geometrically regular curves* of geometric degree d. Any curve Γ_F, $F \in R_d$, is its own resolution (r = identity). By connecting any two $F, G \in R_d$

with a path in R_d, one may construct an isotopy (which may be effected by an ambient isotopy) between the curves Γ_F and Γ_G in $\mathbf{CP}^2$; so all these curves are diffeomorphic, and of the same knot type in the plane. More generally, $F \in Q_d$ lies in a maximal connected subset of Q_d of polynomials G such that Γ_F and Γ_G are ambient isotopic, through algebraic curves. These subsets form a stratification of Q_d which is little understood. Zariski [Z] showed that two (singular) curves in Q_6, homeomorphic and with the same type and number of singularities (cusps), were not in the same stratum, by showing that the knot groups $\pi_1(\mathbf{CP}^2 - \Gamma_F)$ and $\pi_1(\mathbf{CP}^2 - \Gamma_G)$ were not isomorphic. In general, as we will see below, the knot group cannot distinguish strata.

An interesting question (I do not know to whom it is due: I heard of it in Dennis Sullivan's problem seminar at M.I.T. in the summer of 1974) is whether there are curves Γ_F and Γ_G which are ambient isotopic but *not* so through algebraic curves. I know of no results here.

The incidence structure of this stratification of Q_d by "algebraic ambient isotopy types" is, especially, not understood: this is the theory of degenerations. It can be proved that the knot group associated to a given stratum is the homomorphic image of the knot group associated to any stratum incident to the given stratum. Partly, it was the desire to apply this fact to the proof of the Zariski Conjecture (see below) which led investigators for many years to the study of some particular (unions of) strata to which we now turn.

First we recall the two simplest sorts of singularities. A *cusp* has a single branch, locally given by $z = t^2$, $w = t^3$; the link of a cusp is a trefoil knot (of a fixed handedness once one establishes conventions). A *node* has two branches, each itself nonsingular, with distinct tangent lines; it can be locally given by the equation $zw = 0$, and its link is a Hopf link of two components (linking number $+1$). A curve Γ is a *node curve* if all its singularities (if any) are nodes, and a *cusp curve* if all its singularities are either nodes or cusps.

We also recall, what we have not needed before, the notion of reducibility: a curve Γ is *reducible* if its resolution is not connected; alternatively Γ_F is reducible if and only if F is reducible but square-free. A curve that is not reducible is *irreducible.*

The extreme of reducibility is displayed by any $F \in Q_d$ which is the product of d linear factors. Then the curve Γ_F is the union of d projective lines, which we will say (here) are in *general position* if Γ_F is a node curve, that is, if no three of the lines are concurrent. Let $L_d \subset Q_d$ be the set of all such completely reducible curves. Then L_d is a single stratum. Let $N_d \subset Q_d$ be the set of polynomials of node curves; N_d is a union of strata. What is now called the Severi Conjecture is the statement that L_d is incident to every stratum in N_d; in other words, that every node curve can be degenerated to d lines in general position. We will compute the knot group of d lines in general position below. It is, in particular, abelian. Consequently, the truth of the Severi Conjecture would imply that *the knot group of any node curve is abelian*—a statement long known as the Zariski Conjecture, which has recently been proved true by quite other means [F-H, De]. Of course, independent of the truth of the Severi Conjecture, one can study the union $M_d \subset N_d$ of those strata which actually are incident to L_d. Moishezon [Mo] calls M_d the *mainstream* of node curves in his investigation of "normal forms for braid monodromies". Such normal forms (when they exist) enrich the datum of the knot group by giving it in a particularly nice presentation related to the algebraic geometry.

Now let $K_d \subset Q_d$ correspond to the cusp curves. Here the knot groups need no longer be abelian. In fact, for

$$F(z_0, z_1, z_2) = z_1^2 z_2^2 + 4z_0(z_2^3 - z_1^3) + 6z_0^2 z_1 z_2 - 27z_0^4$$

in K_4, a curve with three cusps and no nodes (which has resolution

$$r : \mathbf{CP}^1 \to \Gamma_F : (t_0 : t_1) \mapsto (t_0^2 t_1^2 : t_1^4 + 2t_0^3 t_1 : 2t_0 t_1^3 + t_0^4)),$$

the knot group can be computed (as by Zariski [Z] or, algebraically, by Abhyankar [Ab]) to have the presentation $(a, b : aba = bab, a^4 = 1, a^2 = b^2)$, making it non-abelian of order 12.

The knot groups of cusp curves have been studied because of their application to the study and possible classification of complex (algebraic) surfaces. In fact, if $f : Y \to \mathbf{CP}^2$ is a so-called stable finite morphism, $\Sigma' \subset Y$ the locus where f is not étale, $\Sigma = f(\Sigma')$, then Σ is a cusp curve.

Zariski commissioned van Kampen, in the early 1930's, to calculate the knot group of an arbitrary curve [vK]; van Kampen gave his solution in terms of a certain presentation of the knot group. If Γ has (geometric) degree d, then van Kampen's presentation has d generators $x_1, \ldots, x_d$ which represent loops in a fixed projective line $\mathbf{CP}^1_\infty$ transverse to Γ; the intersection $\Gamma \cap \mathbf{CP}^1_\infty$ contains d points $P_1, \ldots, P_d$, and x_i is a loop from a basepoint $* \in \mathbf{CP}^1_\infty$ out to P_i, around it once counterclockwise, and back to $*$. One relation is then that $x_1 \ldots x_d = 1$. The rest arise by carrying $\mathbf{CP}^1_\infty$ around certain loops of lines. In fact, let $\mathbf{CP}^{2*}$ be the dual projective plane, each point of which is a line in $\mathbf{CP}^2$; and let Γ^* contain all lines which are either tangent to Γ or pass through one of its singular points. Then Γ^* is a curve in $\mathbf{CP}^{2*}$. If $*$ and $\mathbf{CP}^1_\infty$ are sufficiently general, then the pencil of lines in $\mathbf{CP}^2$ through $*$, which is itself a line in $\mathbf{CP}^{2*}$, will be transverse to Γ^*. The (free) fundamental group of the complement of Γ^* in this pencil is naturally represented in the automorphism group of the free group $(x_1, \ldots, x_d : x_1 \ldots x_d$

$= 1$). The rest of the relations needed for the van Kampen presentation of $\pi_1(\mathbf{CP}^2 - \Gamma; *)$ come, then, by declaring this representation trivial. One obtains a finite presentation, of course, by choosing generators of the acting free group; Moishezon's problem of "normal forms" is essentially the problem of making a good choice. Several modernizations [Abe], [Che], [Cha] of van Kampen's proof have been published in recent years.

In a *standard* van Kampen presentation (where the generators of the acting free group are free generators), each relation corresponds either to a singularity of Γ or to a simple vertical tangent to Γ; and (up to the action of the corresponding free generator) each relation is of a certain canonical form, which depends only on the *closed braid type* (§7) of the link of the branch(es) at the point of Γ, through which the line in the pencil passes that gives the relation in question, where this line itself is used to find the axis of the closed braid. In particular, the knot group of a node curve always has a standard van Kampen presentation in which each relation either sets conjugates of two x_i equal (from a simple vertical tangent) or says that two such conjugates commute (from a node); if "conjugates" could be deleted, the Zariski Conjecture would be trivially true.

There is also a great body of work on "knot groups" of curves in (compact, smooth) complex surfaces other than $\mathbf{CP}^2$, and on the related issue of fundamental groups of surfaces; we cannot touch on these topics here.

§6. Global knot theory in brief—the affine case

Little appears to be known about algebraic curves in affine space, from the knot-theoretical viewpoint. The gross algebraic topology (even just homology theory) of $\mathbf{CP}^2$ is implicated with the quite rigid geometry; but affine space is contractible, and on the other hand its geometry is "infinite" (for instance in the sense that there are Lie groups of arbitrarily high dimension contained in the group of biregular automorphisms of $\mathbf{C}^2$), so that the conspirators have fallen out and neither can give away much about the other.

One might think, for example, to study the embedding of a curve Γ in $\mathbf{C}^2$ by first embedding $\mathbf{C}^2$ itself into $\mathbf{CP}^2$. Then the affine complement $\mathbf{C}^2 - \Gamma$ becomes the projective complement $\mathbf{CP}^2 - (\Gamma \cup \mathbf{CP}^1_\infty)$, where $\Gamma \cup \mathbf{CP}^1_\infty$ is a (reducible) projective algebraic curve. The obstacle to this program is the unfortunate fact that $\mathbf{C}^2$, just as an algebraic surface, without distinguished coordinates, is not uniquely embedded as $\mathbf{CP}^2 - \mathbf{CP}^1_\infty$. Any biregular automorphism of $\mathbf{C}^2$ (in particular, one of the vast majority which cannot be extended biregularly to $\mathbf{CP}^2$)