§6. Global knot theory in brief—the affine case

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$= 1$). The rest of the relations needed for the van Kampen presentation of $\pi_1(\mathbf{CP}^2 - \Gamma; *)$ come, then, by declaring this representation trivial. One obtains a finite presentation, of course, by choosing generators of the acting free group; Moishezon's problem of "normal forms" is essentially the problem of making a good choice. Several modernizations [Abe], [Che], [Cha] of van Kampen's proof have been published in recent years.

In a *standard* van Kampen presentation (where the generators of the acting free group are free generators), each relation corresponds either to a singularity of Γ or to a simple vertical tangent to Γ; and (up to the action of the corresponding free generator) each relation is of a certain canonical form, which depends only on the *closed braid type* (§7) of the link of the branch(es) at the point of Γ, through which the line in the pencil passes that gives the relation in question, where this line itself is used to find the axis of the closed braid. In particular, the knot group of a node curve always has a standard van Kampen presentation in which each relation either sets conjugates of two x_i equal (from a simple vertical tangent) or says that two such conjugates commute (from a node); if "conjugates" could be deleted, the Zariski Conjecture would be trivially true.

There is also a great body of work on "knot groups" of curves in (compact, smooth) complex surfaces other than $\mathbf{CP}^2$, and on the related issue of fundamental groups of surfaces; we cannot touch on these topics here.

§6. Global knot theory in brief—the affine case

Little appears to be known about algebraic curves in affine space, from the knot-theoretical viewpoint. The gross algebraic topology (even just homology theory) of $\mathbf{CP}^2$ is implicated with the quite rigid geometry; but affine space is contractible, and on the other hand its geometry is "infinite" (for instance in the sense that there are Lie groups of arbitrarily high dimension contained in the group of biregular automorphisms of $\mathbf{C}^2$), so that the conspirators have fallen out and neither can give away much about the other.

One might think, for example, to study the embedding of a curve Γ in $\mathbf{C}^2$ by first embedding $\mathbf{C}^2$ itself into $\mathbf{CP}^2$. Then the affine complement $\mathbf{C}^2 - \Gamma$ becomes the projective complement $\mathbf{CP}^2 - (\Gamma \cup \mathbf{CP}^1_\infty)$, where $\Gamma \cup \mathbf{CP}^1_\infty$ is a (reducible) projective algebraic curve. The obstacle to this program is the unfortunate fact that $\mathbf{C}^2$, just as an algebraic surface, without distinguished coordinates, is not uniquely embedded as $\mathbf{CP}^2 - \mathbf{CP}^1_\infty$. Any biregular automorphism of $\mathbf{C}^2$ (in particular, one of the vast majority which cannot be extended biregularly to $\mathbf{CP}^2$)

will move Γ around, and so the configuration of $\Gamma \cup \mathbf{CP}^1_\infty$ is not determined by the embedding of Γ in $\mathbf{C}^2$. (For instance, though the geometric number of points at infinity on Γ is determined by Γ, the algebraic intersection number of the closure of Γ with the line at infinity can be made arbitrarily large. Likewise the local singularities at infinity are not determined by the affine curve.)

The main theorems known here have been proved by Abhyankar and his collaborators [A-M, A-S]. They are unknotting theorems, in the sense that they take this form: "Let Γ be a certain curve in $\mathbf{C}^2$, and let $i : \Gamma \to \mathbf{C}^2$ be any algebraic embedding; then there is a biregular automorphism of $\mathbf{C}^2$ returning i to the inclusion map". Briefly, such a curve Γ cannot be knotted in $\mathbf{C}^2$.

However, for most of the curves they deal with, these theorems are not genuinely topological, for the reimbedding i is required to be an embedding *of* Γ *with its given structure as a variety*, and generally there might be moduli. Only in the original theorem [A-M] (which had been stated, but not correctly proved, by Segre) are there no conceivable moduli, when Γ is a straight line. Then the theorem is this.

THEOREM. *Let* $\Gamma \subset \mathbf{C}^2$ *be an algebraic curve without singularities, homeomorphic to* $\mathbf{C}$. *Then there is a biregular change of coordinates* $A : \mathbf{C}^2 \to \mathbf{C}^2$ *so that* $A\Gamma$ *is a straight (complex) line.*

A topological proof has been given in [Ru 4]. It goes like this. One shows (just as for a singular point) that the intersection of Γ (which we can assume to be parametrized by $z = p(t)$, $w = q(t)$, $p, q \in \mathbf{C}[t]$) with a *very large* bidisk boundary is an iterated torus knot $K = O\{m_1, n_1; \ldots; m_s, n_s\}$, with $m_1 = m/GCD(m, n), n_1 = n/GCD(m, n), m = \deg p, n = \deg q$. By hypothesis, K is a slice knot. This forces $K = O$, in particular, one of m_1, $n_1$is 1. Thereafter the argument is as in [A-M]—if (say) $m_1 = 1$ and p and q are monic, then the biregular change of coordinates $(z, w) \mapsto (z, w - z^{m/n})$ carries Γ to another curve satisfying the hypotheses, of lower bidegree; and so we proceed until one of z, w is linear and the other constant.

As to analytic curves in affine space, almost nothing is known. The obvious analogue of the Theorem above is definitely false: for it is known that the unit disk in $\mathbf{C}$ can be properly analytically embedded in $\mathbf{C}^2$ [H]; since the disk and the line are analytically inequivalent, no analytic change of coordinates in $\mathbf{C}^2$ could unknot the disk to a line. It is, however, perfectly possible that every such disk is smoothly unknotted. Presently I am unable even to prove that an analytic line in $\mathbf{C}^2$ is smoothly unknotted.