§6. On classification of Real Algebraic Sets

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Briefly the proof of Theorem 5.3 goes as follows: By a standard argument, $\pi_i(PL/A_k)$ coincides with the concordance classes of A_k-structures on S^i (the exotic A_k-spheres). Since $\pi_i(PL/A) = \varinjlim \pi_i(PL/A_k)$ it follows by definitions that the inclusion $\pi_i(PL/A) \to \eta_i^A$ is an injection, where η_i^A is the cobordism group of i-dimensional A-manifolds. Then we construct a Thom space MA such that $\pi_i(MA) \approx \eta_i^A$ (by using a transversality argument for A-manifolds). Then it turns out that the map $\eta_i^A \to H_i(B_A; \mathbf{Z}/2\mathbf{Z})$ given by $\{M \xrightarrow{\nu_M} B_A\} \mapsto (\nu_M)_* [\mathrm{M}]$ is an injection. We can put these maps into the following commutative diagram:

$$\begin{array}{ccccc} & & \pi_i(PL/A) & \rightarrow & \eta_i^A \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle f} & & \downarrow \\ H_i(PL/A; \mathbf{Z}) & \xrightarrow{r} & H_i(PL/A; \mathbf{Z}/2\mathbf{Z}) & \xrightarrow{g} & H_i(B_A: \mathbf{Z}/2\mathbf{Z}) \end{array}$$

where h is the Hurewicz map, r is the reduction and g is induced by inclusion. Since the other two maps are injections then f must be injection. In fact f is a split injection since it is a map between $\mathbf{Z}/2\mathbf{Z}$-vector spaces. Hence h is a split injection. This implies that all k-invariants of PL/A is zero, i.e. PL/A is a product of Eilenberg-Mclaine spaces $\prod K(\mathbf{Z}/2\mathbf{Z}, n_i)$. Then by dualizing the split injection $g \circ f$ we get a surjection

$$H^i(B_A; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\lambda} \mathrm{Hom}(\pi_i(PL/A); \mathbf{Z}/2\mathbf{Z})$$

Let $\delta_{n_i} \in H^{n_i}(B_A; \mathbf{Z}/2\mathbf{Z})$ such that $\lambda(\delta_{n_i})$ is the generator of $\mathbf{Z}/2\mathbf{Z}$.

$$\delta = \prod \delta_{n_i} \text{ defines a map } B_A \to \prod_i K(\mathbf{Z}/2\mathbf{Z}, n_i) = PL/A\,.$$

Then the map $\pi \times \delta\colon B_A \to B_{PL} \times PL/A$ turns out to be the desired splitting. The calculation of ρ_n can be done by using the geometric interpretation of $\pi_*(PL/A)$.

The set $\mathscr{S}_A(M) = \bigoplus_n H^n(M; \pi_n(PL/A))$ measures the number of different "topological resolutions" of M, up to concordance (i.e. A-structures). Therefore often $\mathscr{S}_A(M)$ is infinite; and $\mathscr{S}_A(M^8)$ has 2^{26} elements for any closed 8-manifold M^8.

§6. On classification of Real Algebraic Sets

The resolution and complexification properties of real algebraic sets impose many restrictions on the underlying topological spaces. To give a topological characterization of algebraic sets one has to find all such properties, such that a set is homeomorphic to an algebraic set if and only if it satisfies these properties. Call a polyhedron V an *Euler space* if $\chi(\text{Link}(x))$ is even for all vertices $x \in V$. Recall that all algebraic sets are Euler spaces, in fact in low dimensions this topological property completely determines compact algebraic sets (and hence all algebraic sets by Proposition 3.1).

THEOREM 6.1. *Let X be a compact polyhedron of dimensions $\leqq 2$. Then X is homeomorphic to a real algebraic set if and only if X is an Euler spaces.*

This theorem was announced in [AK_2] and a proof was given [AK_7]. Since [AK_7] did not appear in print we repeat that proof here. This proof is very useful to understand the high dimensional case. It is done by first constructing a "topological resolution" for X then proceeding as in the proof of Theorem 5.1.

Proof: The proof of case $\dim(X) \leqq 1$ follows from Theorem 4.1, so assume that $\dim(X) = 2$. Let X' be the barycentric subdivision of X. Let X_i = the i-skeleton of X'. Then (exercise) X_1 satisfies the even local Euler characteristic condition also. We will say a one simplex in X' has type i ($i = 0,1$) if the number of faces containing it is congruent to $2i$ mod 4. Let X_{1i} be the unions of edges of type i, then (exercise) X_{10} and X_{11} each satisfy the even local Euler characteristic condition. Hence, they have resolutions $\pi_{1i} : Z_{1i} \to X_{1i}$ where Z_{1i} are unions of circles, and the π_{1i} are diffeomorphisms over $X_{1i} - X_0$.

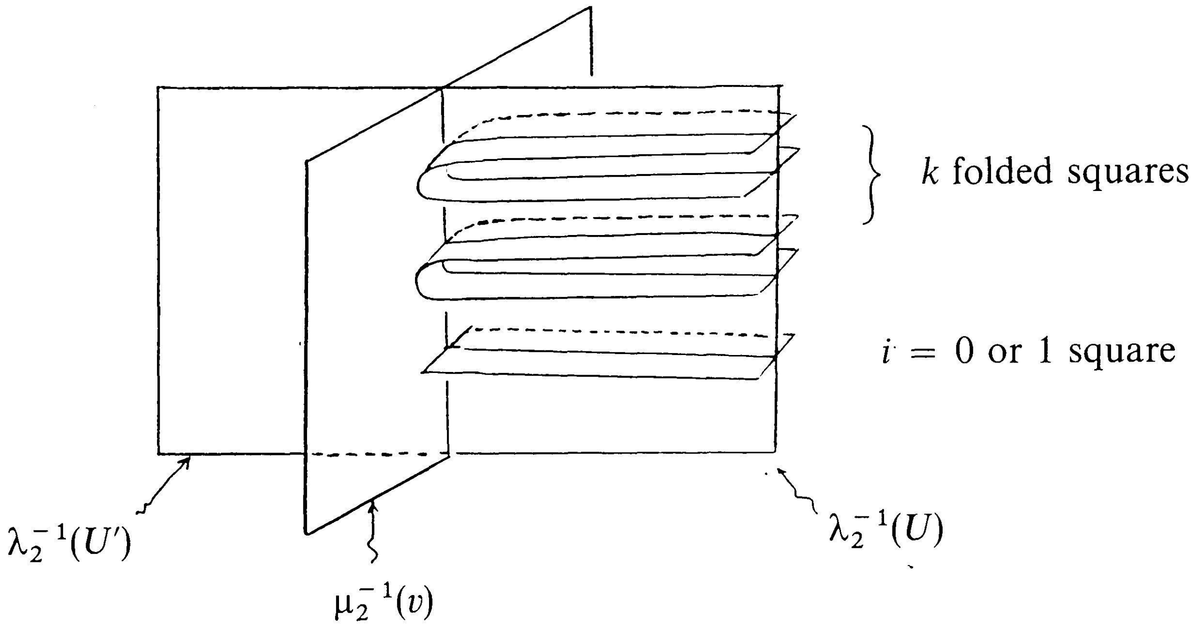
First, we imbed X_0 in $\mathbf{R}^4$. Now let $V_1 = B(\mathbf{R}^4, X_0)$ and let $\mu_1 : V_1 \to \mathbf{R}^4$ be the projection. We may imbed $Z_{10} \cup Z_{11}$ in V_1 so that $\mu_1(Z_{1i}) \cup X_0$ is homeomorphic to X_{1i} and $\mu_1|_{Z_{1i}} = \pi_{1i}$. Since V_1 has totally algebraic homology, by Theorem 2.8 we may assume after replacing V_1 by $V_1 \times \mathbf{R}^n$ that each component of each Z_{1i} is a nonsingular algebraic subset of V_1. We now let $V_2 = B(V_1, Z_{10} \cup Z_{11})$ and $\lambda_2 : V_2 \to V_1$ be the projection and $\mu_2 : V_2 \to \mathbf{R}^4$ be the composition of μ_1 and λ_2. We will now imbed a surface Z_2 in V_2 so that

$$\mu_2(Z_2) \cup \mu_1(Z_{10} \cup Z_{11}) \cup X_0$$

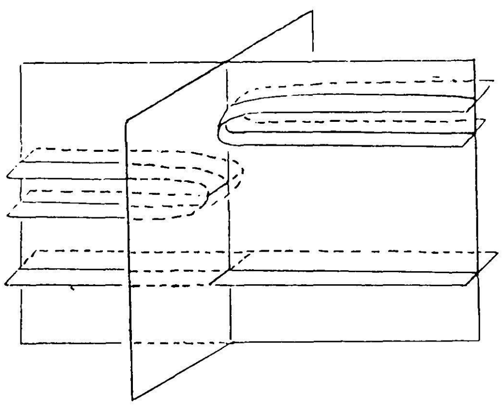
is homeomorphic to X.

We pick some pairing of the faces coming into each edge, i.e. there are an even number of them, and we divide them into groups of two. This gives a resolution of $X - X_0$, namely, take the disjoint union of the faces with vertices deleted and identify two edges if they are in the same group of two. This will be part of our surface Z_2, but we will not imbed it until later. We will first imbed the part of Z_2 lying over a small neighborhood of X_0.

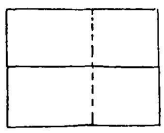
Take any vertex v of X_0 and let e be an edge containing v, let $i = 0, 1$ be such that $e \subset X_{1i}$. Then $e = \mu_1(U)$ for some interval U in Z_{1i}. Let there be $4k + 2i$ faces containing e. Pick a point p in $\mu_2^{-1}(v) \cap \lambda_2^{-1}(u)$ where $u \in U$ is the point so that $\mu_1(u) = v$. Then in a neighborhood of p, we have two codimension one submanifolds $\mu_2^{-1}(v)$ and $\lambda_2^{-1}(Z_{1i})$. We imbed $k + i$ squares in a neighborhood of p as indicated below.



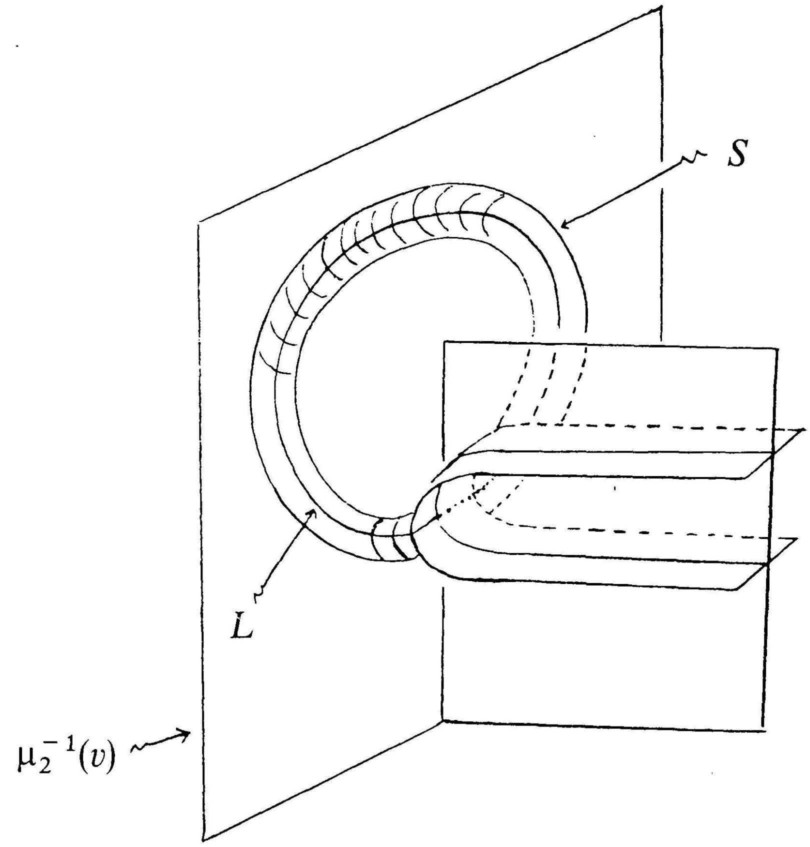
We do this for each edge containing v. Notice that one of these edges is $\mu_1(U')$ for some interval U' in Z_{1i} so $U' \cap U = u$, i.e. the interval on the other side of u. If $i = 1$, we connect the bottom squares of the two sides together as shown below.



In the end, we have a bunch of squares

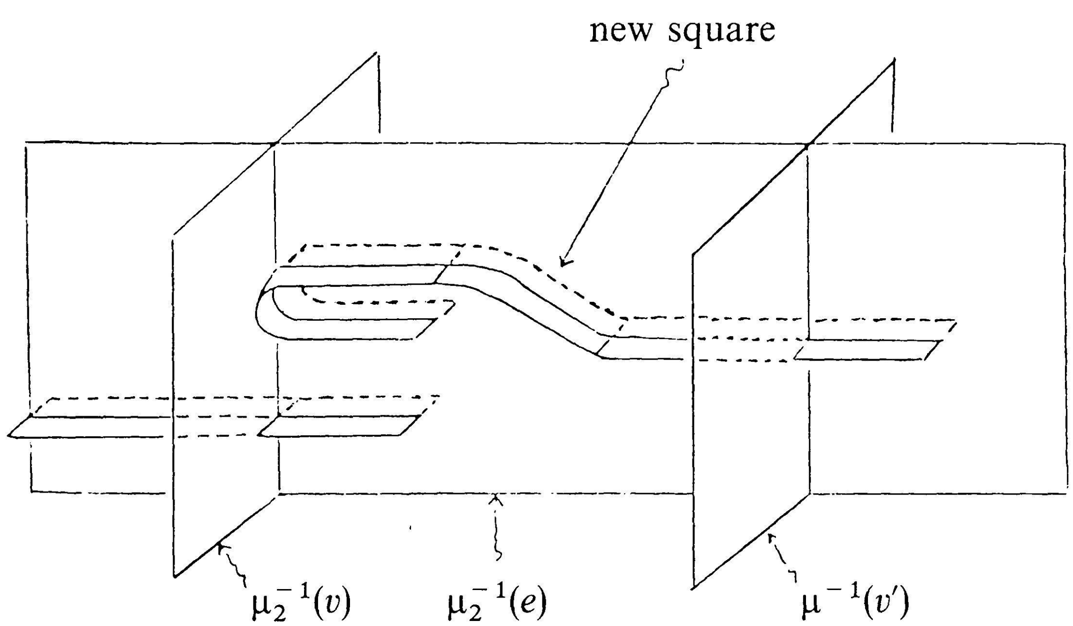


whose horizontal midlines are mapped by μ_2 to v and whose vertical midlines are mapped by λ_2 to $Z_{10} \cup Z_{11}$. Furthermore, this map is either equivalent to x^2 or x if we choose our imbedding nicely. To each corner of each square, we may assign a face of X' which contains v so that the following conditions are met: each face containing v is assigned to exactly two corners, if e is the edge containing μ_2 of the top half of the vertical midline, then the faces assigned to the top two corners each contain e and are, in fact, paired, and likewise, for the bottom two corners and the bottom midline half. We may now form a number of polygons by taking the vertical side edges of all the squares and identifying their endpoints, if the corresponding faces are the same. We claim these polygons are the boundary of a surface S which contains L, a union of arcs and circles in general position so that S is a regular neighborhood of L, $\partial S \cap L$ is the union of the endpoints of all the arcs in L and $\partial S \cap L$ is also the union of all the midpoints of the sides of the boundary polygons.



Given this, we imbed S in V_2 so that S misses $\lambda_2^{-1}(Z_{10} \cup Z_{11})$ and $\mu_2^{-1}(X_0 - v)$ and so $\mu_2^{-1}(v) \cap S = L$, and so S intersects the squares we have already imbedded in the union of the side edges of all the squares, furthermore, these intersect in the natural way so that the point of $L \cap \partial S$ which corresponds to the midpoint of a side of a polygon, is mapped to the midpoint of the corresponding side of a square. So, letting S' be S union all the squares, we have that $\mu_2(S')$ is homeomorphic to the star of v in the union of the faces of X. This is because clearly $\mu_2(S')$ is the cone on $\mu_2(\partial S')$, but $\mu_2(\partial S')$ is obtained by taking the polygon formed by all the top and bottom sides of the squares and identifying endpoints corresponding to the same face and identifying midpoints of all sides which map to the same edge of X'. This is clearly the link of v in the closure of all faces.

We do this for all the vertices and we get a surface S''. We now add some more squares. For each edge e of X', let v and v' be its vertices. We have previously paired up the faces containing e. For each pair of faces, we have a corresponding top or bottom side of a square over v, and a top or bottom side of a square over v' (namely the sides between the two corners assigned to the pair), we connect these two sides with another square as shown (S is not shown).



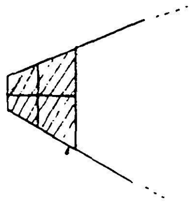
If we do this for each pair of faces coming into each edge of X', we get a surface S^* imbedded in V_2 so that $\mu_2(S^*)$ is homeomorphic to a neighborhood of X_1 in the union of the faces of X'. It is now easy to imbed a bunch of discs (one for each face of X') and so get a surface Z_2 in V_2, so that $\mu_2(Z_2)$ is the union of the faces of X' and so

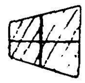
$$\mu_2(Z_2) \cup \mu_1(Z_{10} \cup Z_{11}) \cup X_0$$

is homeomorphic to X.

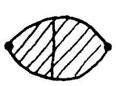
We could now try to approximate Z_2 by a nonsingular algebraic set and then blow down to finish off the proof, but the problem is Z_2 is not stable, i.e. Z_2 is not transverse to $\mu_2^{-1}(X_0)$. However, we may, after replacing V_2 by $V_2 \times \mathbf{R}^k$, assume that $Z_2 \cap \mu_2^{-1}(X_0)$ is a union of nonsingular algebraic sets. An exercise below shows that if we blow up along each of these algebraic sets twice, then Z_2 becomes transverse to $\mu_2^{-1}(X_0)$. Then we are able to finish off by approximating Z_2 by an algebraic set (Theorem 2.8) and blowing down, first over $Z_{10} \cup Z_{11}$ and then over X_0 (Proposition 3.3).

We deferred the proof that the polygon bounds the surfaces S, so we give it here. First, by induction, we may assume all polygons have either one or two sides, for we may take three sides and fill in part of the surface and reduce to the problem with those three sides replaced by one side (see below).

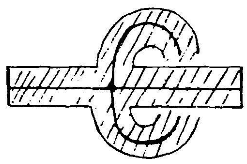


The shaded region is filled in part, $+$ is part of L. If we can fill in the rest, then adding on  will fill in all of it.

But we can easily fill in a polygon with two sides, and we can also fill in two one sides. Since the total number of sides is even, we are done.



two sides filled in



two one-sides filled in

Exercise: Think of $\mathbf{R}^n$ as $\{(x, y, z, w) \mid x, y, z \in \mathbf{R} \text{ and } w \in \mathbf{R}^{n-3}\}$. Let $S = \{z = x^a y^b, w = 0\}$ and $T = \{z = 0\}$. Blow up along the x axis twice and along the y axis twice, and show that after blowing up S becomes transverse to the inverse image of T, (assuming $a = 1, 2$ and $b = 1$ or 2). Note that by imbedding the S in the above proof correctly, we may assume that locally it looks like this with $T = \mu_2^{-1}(v)$. □

The proof of the 2-dimensional case is done by first constructing an appropriate topological resolution. In the general case this leads us to make the following definition. A *topological resolution tower* $\{V_i, V_{ji}, p_{ji}\}$ is a collection of smooth manifolds V_i, $i = 0, \ldots, n$, subsets $V_{ji} \subset V_i$, $j = 0, \ldots, i - 1$ and maps $p_{ji}: V_{ji} \to V_j$ satisfying the following properties:

(I) $p_{ji}(V_{ji} \cap V_{ki}) \subset V_{kj}$ for $k < j < i$.

(II) $p_{kj} \circ p_{ji}|_{V_{ji} \cap V_{ki}} = p_{ki}|_{V_{ji} \cap V_{ki}}$ for $k < j < i$.

(III) $p_{ji}^{-1}(\bigcup_{m \leq k} V_{mj}) = V_{ji} \cap \bigcup_{m \leq k} V_{mi}$.

(IV) V_{kj} is a union of codimension one smooth submanifolds of V_j in general position; we call them the sheets of V_{kj}. If S is a sheet of V_{kj} then $p_{ji}^{-1}(S)$ is the intersection of V_{ji} with a union of sheets of $\bigcup_{m \leq k} V_{mi}$.

(V) p_{ji} is smooth on each sheet of V_{ji}, and

$$p_{ji} : V_{ji} - \bigcup_{k<j} V_{ki} \to V_j - \bigcup_{k<j} V_{kj}$$

is a locally trivial fibration.

(VI) For any $q \in V_{ji}$ let $q_i = q$, $q_j = p_{ji}(q)$.

Then there are smooth local coordinates

$$\theta_a : (U_a, 0) \xrightarrow{\approx} (V_a, q_a),\ a = i, j,$$

where U_a is an open neighborhood of 0 in some $\mathbf{R}^{c_{a0}} \times \mathbf{R}^{c_{a1}} \times \ldots \times \mathbf{R}^{c_{aa}}$ such that:

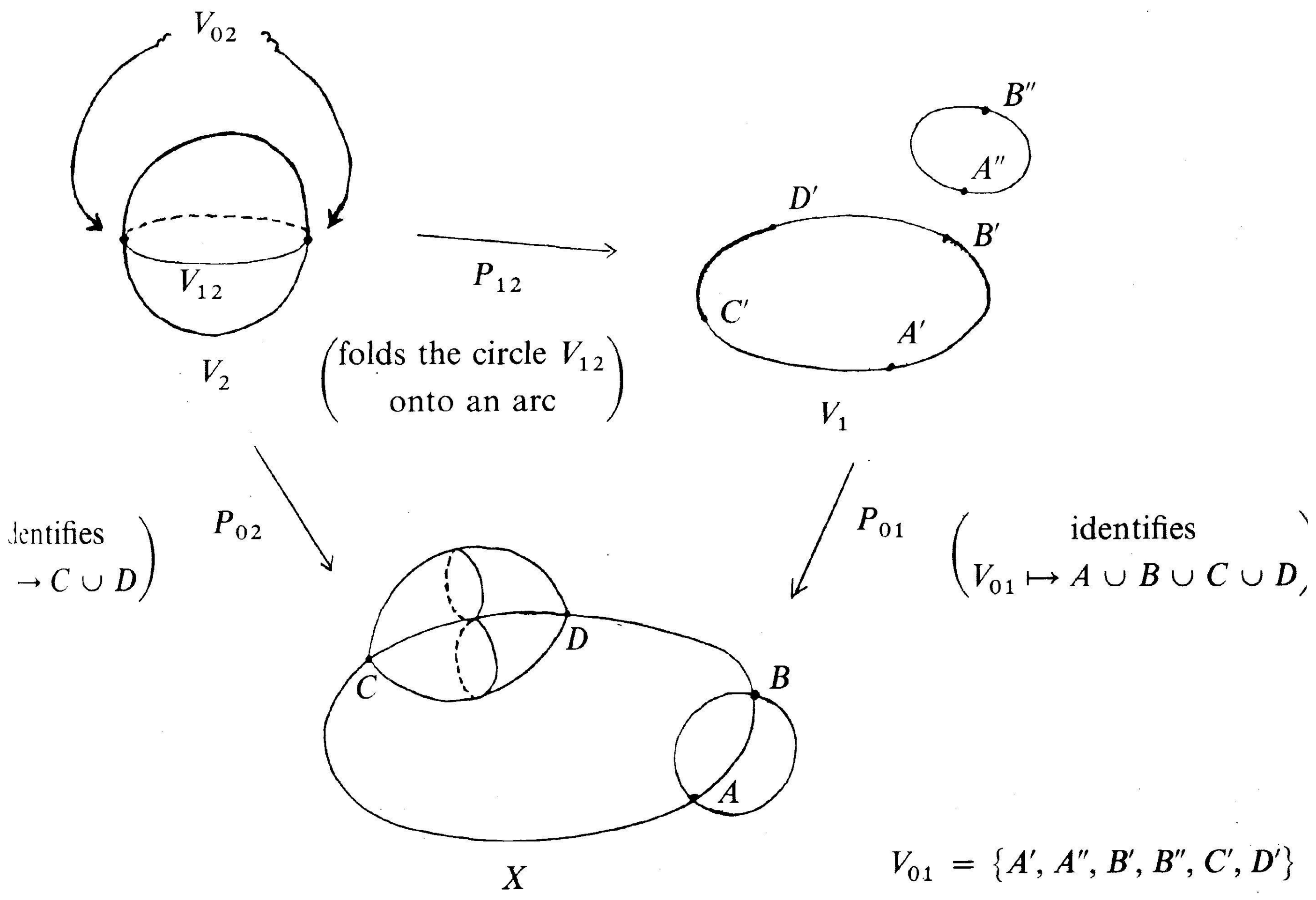
$$(1)\ \theta_a^{-1}(V_{ta}) = \begin{cases} \emptyset & \text{if } c_{at} = 0, \\ \{x \mid \prod_{s=1}^{c_{at}} x_{ts} = 0\} \cap U_a & \text{if } c_{at} \neq 0, \end{cases}$$

$$(2)\ [\theta_j^{-1} \circ p_{ji} \circ \theta_i(x)]_{km} = \prod_{t=0}^{k} \prod_{s=1}^{c_{it}} x_{ts}^{I^{ts}_{km}} \cdot \varphi_{km}(x) \quad \text{if } k < j,$$

where I^{ts}_{km} is a nonnegative integer, and each φ_{km} is a nowhere zero smooth function. x_{ts} denotes the s-th coordinate of x in $\mathbf{R}^{c_{it}}$, and $[\theta_j^{-1} \circ p_{ji} \circ \theta_i(x)]_{km}$ denotes the m-th coordinate of $\theta_j^{-1} \circ p_{ji} \circ \theta_i(x)$ in $\mathbf{R}^{c_{jk}}$.

Even though (VI) looks like an algebraic condition it is a topological condition. It says that topologically the map p_{ji} has only certain types of singularities (i.e. it folds or crushes). We call a topological resolution tower $\{V_i, V_{ji}, p_{ji}\}$ *an algebraic resolution tower* if all V_i, V_{ji} are compact algebraic sets and p_{ij} are entire rational functions.

The realization $|\mathcal{T}|$ of a (topological or algebraic) resolution tower $\mathcal{T} = \{V_i, V_{ji}, p_{ji}\}$ is the quotient space $\bigcup V_i/x \sim p_{ji}(x)$ for $x \in V_{ji}$. $|\mathcal{T}|$ is a stratified space with i-th stratum equal to $V_i - \bigcup_{j<i} V_{ji}$. It turns out that if $\mathcal{T}$ is an algebraic resolution tower then $|\mathcal{T}|$ is an algebraic set. $|\mathcal{T}|$ is a generalization of an A-space.



Real algebraic sets are obvious candidates for realizations of topological resolution towers: If X is a real algebraic set, it has an algebraic stratification

$$X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X$$

with $\mathrm{Sing}(X_i) \subset X_{i-1}$, $i = 1, \dots, n$. Then the resolution of singularities theorem [H] says that there is a multiblowup:

$$V_n = Z_n \xrightarrow{\pi_n} Z_{n-1} \to \dots \to Z_1 \xrightarrow{\pi_1} Z_0 = X$$

with $\pi_1 : Z_1 \to Z_0$ is a multiblowup of X which resolves the singularities of X_1, i.e. there is a nonsingular $V_1 \subset Z_1$ making the following commute

$$\begin{array}{ccc} V_1 & \hookrightarrow & Z_1 \\ \downarrow & & \downarrow \pi_1 \\ X_1 & \hookrightarrow & Z_0 \end{array}$$

If $\pi_{ji} : Z_i \to Z_j$ is the composition projection, then π_{i+1} is a multiblowup of Z_i which resolves the singularities of the strict preimage of X_{i+1} under π_{0i}, i.e. there is a nonsingular $V_{i+1} \subset Z_{i+1}$ and the commutative diagram

$$\begin{array}{ccc} V_{i+1} & \hookrightarrow & Z_{i+1} \\ \downarrow & & \downarrow {\scriptstyle 0,\, i+1} \\ X_{i+1} & \hookrightarrow & Z_0 \end{array}$$

Let $V_{ji} = \pi_{ji}^{-1}(V_j) \bigcap V_i$ and $\pi_{ji}|_{V_{ji}} = p_{ji} : V_{ji} \to V_j$. Then one can show that

$$X \approx \bigcup V_i / p_{ji}(x) \sim x\,,$$

for $x \in V_{ji}$.

In fact after refining this process one gets:

THEOREM 6.2. *A set is an algebraic set if and only if it is homeomorphic to a realization* $|\mathscr{T}|$ *of some algebraic resolution tower* $\mathscr{T} = \{V_i, V_{ji}, p_{ji}\}$.

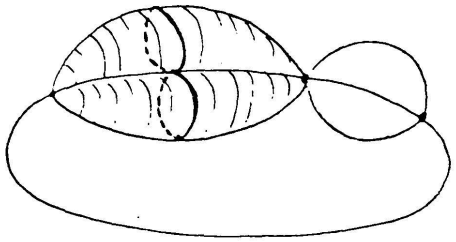
Hence we have natural maps

$$\begin{array}{ccc} \left\{\text{Algebraic sets}\right\} & \xrightarrow{\tau} & \left\{\begin{array}{l}\text{realization of topological} \\ \text{resolution towers}\end{array}\right\} \\ {\scriptstyle \text{onto}} \searrow & & \nearrow {\scriptstyle \rho} \\ & \left\{\begin{array}{l}\text{realization of algebraic} \\ \text{resolution towers}\end{array}\right\} & \end{array}$$

where ρ is the forgetful map, and τ is the composition. We will denote the set of realization of topological resolution towers by $\mathscr{R}$. To characterize algebraic sets topologically, we need to show that ρ maps onto $\mathscr{R}$. Presently to prove this we need each V_i to be diffeomorphic to a nonsingular algebraic set with totally algebraic homology (see §2). We believe that these restrictions should not be necessary.

Once surjectivity of τ is proven, then it would be useful to find the combinatorial conditions which characterize elements of $\mathscr{R}$ (i.e. algebraic sets). For spaces of dimension $\leqq 2$ the only condition is that the space has to be an

Euler space (Theorem 6.1). In dimension 3 this is not sufficient. For example if X^3 is the suspension $\Sigma(Y^2)$ of Y^2 where

Y^2 =  = Σ (figure 8) $\cup$ Σ (three points) $\cup$ an arc

then X^3 is an Euler space but it can not be in $\mathscr{R}$; in particular X^3 can not be homeomorphic to an algebraic set (also see [K_2] for a discussion of this).

In general we start with a Thom stratified space X, by refining the stratification we can assume that each stratum has a trivial normal bundle. Then by proceeding as in [Su_2] we can find obstructions $\alpha_k \in H^k(X(k); \Gamma_{n-k-1})$ to X being an algebraic set with this stratification, where $X(k)$ is the k-th stratum of X, $n = \dim(X)$ and Γ_i is the cobordism group of i-dimensional elements of $\mathscr{R}$. For example we can show $\Gamma_0 = \Gamma_1 \cong \mathbf{Z}/2\mathbf{Z}$ and $\Gamma_2 \cong (\mathbf{Z}/2\mathbf{Z})^{16}$. It would be useful to compute the cobordism groups Γ_* for $* \geqq 3$ or reduce the computation to a certain homotopy group of a universal space (as in the smooth cobordism group). A more precise discussion of this section will appear in [AK_9].