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**Kapitel:** 2. A BRIEF RESUME OF DIVISORS ON CURVES

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$$\sigma((\alpha_1, ..., \alpha_r), (\beta_1, ..., \beta_s)) = (\alpha_1\beta_1, ..., \alpha_1\beta_s, \alpha_2\beta_1, ..., \alpha_2\beta_s, ..., \alpha_r\beta_s)$$

is a projective morphism establishing an isomorphism between $\mathbf{P}_{r-1} \times \mathbf{P}_{s-1}$ and the image $\mathscr{S} = \sigma(\mathbf{P}_{r-1} \times \mathbf{P}_{s-1})$. [4, Ex I.2.14] Once I label the coordinates of $\mathbf{P}_{rs-1}$ as $(z_{11}, ..., z_{1s}, z_{21}, ..., z_{2s}, ..., z_{rs})$, $\mathscr{S}$ can be identified with the algebraic subset of $\mathbf{P}_{rs-1}$ cut out by the polynomials

$$\{z_{ij}\, z_{pq} - z_{iq}z_{pj} \mid 1 \leqslant i, p \leqslant r \text{ and } 1 \leqslant j, q \leqslant s\}.$$

$\mathscr{S}$ is an algebraic subvariety of $\mathbf{P}_{rs-1}$, of dimension $r + s - 2$.

In $\mathbf{P}_{rs-1}$ we can also consider the algebraic subvariety $\mathscr{T}$ cut out by the polynomials $\{\sum_{ij} z_{ij}\lambda_k^{ij} \mid 1 \leqslant k \leqslant t\}$. Since $\mathscr{T}$ is cut out by $t \leqslant r + s - 2$ equations and $\dim \mathscr{S} = r + s - 2$, $\mathscr{S}$ and $\mathscr{T}$ have a nonempty intersection, all of whose components have dimension at least $(r+s-2) - t$, which is $\geqslant 0$. [4, p. 48] However, any intersection point of $\mathscr{S}$ and $\mathscr{T}$ corresponds to a pair of points $(\alpha_1, ..., \alpha_r) \in \mathbf{P}_{r-1}, (\beta_1, ..., \beta_s) \in \mathbf{P}_{s-1}$ satisfying (*). The corresponding points $a = \Sigma\, \alpha_i a_i \in A$, $b = \Sigma\, \beta_j b_j \in B$ are nonzero, yet $\varphi(a, b) = 0$. Since this contradicts the bi-injectivity of $\varphi$, I have shown that

$$\dim C \geqslant r + s - 1. \qquad \square$$

The assumption that $K$ is algebraically closed was only needed to guarantee that $\mathscr{S} \bigcap \mathscr{T}$, which by dimension theory corresponds locally to a proper ideal, was nonempty. Hilbert's Nullstellensaltz shows that any proper ideal in a polynomial ring over an algebraically closed field cuts out at least one point.

## 2. A BRIEF RESUME OF DIVISORS ON CURVES

In this section, I will establish notation for divisors, and state the Riemann-Roch theorem. Let $C$ be a nonsingular projective algebraic curve defined over an algebraically closed field $K$. $C$ is contained in some projective space $\mathbf{P}_N$ over $K$, and a (closed) *point* of $C$ is any closed point $(p_0, ..., p_N)$ of $\mathbf{P}_N$ at which all the polynomials cutting out $C$ vanish. The *group of divisors on* $C$ is the free abelian group generated by the points of $C$. Any divisor can be written in the form

$$N = \Sigma\, n_P \cdot P$$

where the $n_P$ are integers, almost all zero. The *degree* of $N$ is the integer $\deg N = \Sigma\, n_P$. The divisor $N$ is *effective* if all the $n_P$ are $\geqslant 0$; this is written as $N \succ 0$. I write $D \succ E$ to mean $D - E \succ 0$.

To any function $f$ on $C$ one can associate a divisor $(f) = \Sigma \operatorname{ord}_P(f) \cdot P$, where $\operatorname{ord}_P(f)$ is the order of zero or pole of $f$ at $P$. For any function $f$, the divisor $(f)$ has degree 0. The divisors $D$, $E$ are *linearly equivalent*, denoted by $D \sim E$, if for some function $f$, $D - E = (f)$. To a divisor $D$ on $C$ one can associate a set of functions on $C$,

$$L(D) = \{\text{functions } f \text{ on } C \mid (f) + D \succ 0\} \bigcup \{0\}\,.$$

Then $L(D)$ is a $K$-vector space of dimension $l(D)$; the set $\mid D \mid = \{$divisors $E \sim D \mid E \succ 0\}$ of the divisors $(f) + D$ corresponding to functions $f$ in $L(D)$ is the *linear system* associated to $D$. If $\{f_0, ..., f_n\}$ is a basis of $L(D)$, then $\mid D \mid$ can be identified with $\mathbf{P}_n$ by associating the divisor

$$(a_0 f_0 + ... + a_n f_n) + D$$

to the triple $(a_0, ..., a_n)$; one writes $\dim \mid D \mid$ for the dimension of this projective space. To define $\dim \mid D \mid$ intrinsically, notice that $\dim \mid D \mid \geqslant r$ if and only if, for all points $P_1, ..., P_r$ in $C$, there is a divisor $E$ in $\mid D \mid$ of the form $E = P_1 + ... + P_r + Q$, with $Q$ effective. Any such divisor $E$ is necessarily effective and linearly equivalent to $D$, and has support containing each $P_i$. (In fact, since $\dim \mid D \mid \geqslant r$ there is a linearly independent set $\{f_0, ..., f_r\}$ of functions in $L(D)$. One can choose $E$ of the form $E = D + (\alpha_0 f_0 + ... + \alpha_r f_r)$ for some $\alpha_0, ..., \alpha_r \in K$.)

If $D \sim E$, then $\mid D \mid = \mid E \mid$, so $\dim \mid D \mid = \dim \mid E \mid$, and $L(D)$ is isomorphic to $L(E)$. Since for any function $f$ on $C$ $\deg(f) = 0$, also $\deg D = \deg E$. In particular, if $\deg D < 0$ then $\mid D \mid$ is empty, and $L(D) = (0)$.

*Definition.* The curve $C$ *admits a* $g_n^r$ if there exists a divisor $D$ on $C$ of degree $n$, and with $\dim \mid D \mid = r$. We call $\mid D \mid$ *the* $g_n^r$ *defined by D.*

Notice that if $D$ defines a $g_n^r$ and $E \sim D$, then $E$ defines the same $g_n^r$. Yet a curve may admit several distinct $g_n^r$'s if it contains non-linearly equivalent divisors all defining $g_n^r$'s. To explain the notation, assume that $L(D)$ has basis $(f_0, ..., f_r)$. Then the map

$$P \to (f_0(P), ..., f_r(P))$$

is a rational map from $C$ into $\mathbf{P}_r$, defined except at the common zeros of all the $f_i$ (the "fixed points" of $\mid D \mid$); via this map, the pullback of every hyperplane in $\mathbf{P}_r$ is a divisor on $C$ of degree $n$. [4, II: 7.7 and 7.8.1]

The Riemann-Roch Theorem defines for each curve two invariants—a nonnegative integer $g$, the *genus*, and a divisor $\mathscr{K}$, the *canonical divisor* (determined only up to linear equivalence). [For a modern proof, cf. 4, Ch. IV.1; an elementary proof is given in 2].

THEOREM (Riemann-Roch). *Let* $C$ *be a projective nonsingular algebraic curve. The genus of* $C$ *is a nonnegative integer* $g$. *For all divisors* $D$ *on* $C$,

$$\dim | D | \geqslant \deg D - g .$$

*If the strict inequality holds,* $D$ *is* special. *For all special divisors* $D$,

$$\dim | D | = \deg D + 1 - g + \dim | \mathscr{K} - D | .$$

COROLLARY. $\deg \mathscr{K} = 2g - 2$; $\dim | \mathscr{K} | = g - 1$; *and all divisors* $D$ *of degree* $> 2g - 2$ *are nonspecial.*

## 3. CLIFFORD'S THEOREM — THE ELEMENTARY PROOF

Clifford's Theorem complements Riemann-Roch by providing information about special divisors, which of necessity are of small degree. The theorem also gives a sufficient condition that the curve $C$ is hyperelliptic. (The theorem owes its name to the appearance of its first part in [1].) The proof I give here is elementary; more typical modern proofs [e.g. 4, Ch. IV, section 5 and 3, Ch. 2, section 3] involve considering whether the canonical morphism $C \to \mathbf{P}_{g-1}$ defined by the canonical divisor $\mathscr{K}$ is an embedding.

*Definition.* $C$ is a *hyperelliptic* curve if its genus $g$ is at least 2, and if $C$ admits a $g^1_2$.

*Remarks.*

1. $C$ is hyperelliptic if and only if there is a rational map $C \to \mathbf{P}_1$ of degree 2.

2. This happens if and only if $C$ has an (affine) equation of the form $y^2 = f(x)$.

3. Part (3) of Clifford's Theorem shows that a hyperelliptic curve has a unique $g^1_2$. Contrast this to the case of an elliptic curve, where $g = 1$. Here any divisor of degree 2 defines a $g^1_2$. Yet choosing distinct points $P$, $Q$ one sees easily that the divisors $2P$ and $P + Q$ are not linearly equivalent, and so define distinct $g^1_2$'s.

THEOREM (Clifford). *Let* $C$ *be a curve of genus* $g$, *and let* $D$ *be an effective special divisor on* $C$. *Then*