Zeitschrift: L'Enseignement Mathématique

Band: 30 (1984)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LARGE FREE GROUPS OF ISOMETRIES AND THEIR GEOMETRICAL USES

Kapitel: §2. The Main Theorem

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DOI: https://doi.org/10.5169/seals-53829

Finally, in § 9 we discuss what can be done in $\mathbf{R}^2$ if we allow area-preserving linear or affine transformations instead of just isometries.

Proofs of several of the results mentioned or used in this paper, such as Theorems 3, 4 (b) and (c), 5, 6, and 7, may be found in [41].

We thank W. Barker, A. Borel, A. Durfee, R. Riley, and D. Sullivan for informative discussions and correspondence on hyperbolic n-space.

§ 2. The Main Theorem

The action of a group, G, on a set, X, is called *fixed-point free* if $g(x) \neq x$ for all $x \in X$ and $g \in G \setminus \{I\}$ (I is the identity of G). The action is called *locally commutative* if, for each $x \in X$, $\{\sigma \in G : \sigma(x) = x\}$ is a commutative subgroup of G; equivalently, if two elements of G have a common fixed point in X then they commute. For any group G and any abstract (reduced) group word w in m variables, the function $f_w : G^m \to G$ is defined by $f_w(\sigma_1, ..., \sigma_m) = w(\sigma_1, ..., \sigma_m)$.

If X is a metric space and also an oriented manifold, then $G(X)$ denotes the group of orientation-preserving isometries of X, with its natural topology. In particular, $G(S^n) = SO_{n+1}$, $G(H^2) = PSL_2(\mathbf{R})$ and $G(H^3) = PSL_2(\mathbf{C})$.

A set in a complete metric space is called *perfect* if it is nonempty, closed and without isolated points; a perfect set has at least $2^{\aleph_0}$ elements.

Theorem 1.

(a) *Each of the groups* $G(S^n)$, *where* n *is odd and* $n \geqslant 2$, $G(\mathbf{R}^n)$, *where* $n \geqslant 3$, *and* $G(H^n)$, *where* $n \geqslant 3$, *has a free subgroup with a perfect set of free generators whose action on the space is fixed-point free.*

(b) $G(H^2)$ *has a discrete free subgroup of rank* 2 (*and hence also rank* $\aleph_0$) *which is fixed-point free, but no such free subgroup of* $G(H^2)$ *can have uncountable rank.*

(c) $G(H^2)$ *and each of the groups* $G(S^n)$, $n \geqslant 2$, *have locally commutative free subgroups with a perfect set of free generators.*

The above theorem is false in all omitted dimensions. This is because the isometry groups in the low dimensions are all solvable, and hence contain no free subgroup of rank 2. Also, each element of SO_{2n+1} has a fixed point on S^{2n}, and this is why part (a) fails for spheres of even dimension. Of course, an uncountable subgroup of $G(H^n)$ (or any infinite subgroup of the compact group SO_n) cannot be discrete. Moreover, $G(\mathbf{R}^n)$ has the Abelian (and therefore amenable) group of translations as a closed normal subgroup, and the quotient is the compact (and therefore amenable as a topological group) group SO_n. It follows that $G(\mathbf{R}^n)$ is an amenable topological group, whence it has no discrete free subgroup of rank two (see [12, § 2]).

The results of Theorem 1 are known except for part (a) for H^3 and S^n if $n \equiv 1 \pmod 4$, and part (c) for S^4. The history of the known cases is the following. The earliest results on free isometry groups are due to Klein and Fricke [16] and Hausdorff [14]. The former showed that $PSL_2(\mathbf{Z})$ is isomorphic to the free product $\mathbf{Z}_2 * \mathbf{Z}_3$, (see [17, Appendix]), whence $PSL_2(\mathbf{R})$, which is isomorphic to $G(H^2)$, contains a free subgroup of rank 2. Since the entire action of $PSL_2(\mathbf{R})$ on H^2 is locally commutative (see § 6), this yields part (c) for H^2 and rank 2. Hausdorff showed that $\mathbf{Z}_2 * \mathbf{Z}_3$ also appears as a subgroup of SO_3. Again, the action of the rotation group SO_3 on S^2 is locally commutative, so this yields part (c) for S^2 and rank 2. This was the foundation of Hausdorff's theorem that there is no finitely additive, rotation-invariant measure defined for all subsets of S^2 and having total measure one. It also forms the basis of the Banach-Tarski paradoxical decomposition of a sphere. It was not until much later, however, that the advantages of local commutativity for such constructions were recognized [34, 13]. The simplest proof that $\mathbf{Z}_2 * \mathbf{Z}_3$ embeds in SO_3 may be found in [33]. However, see [10] for a beautiful proof using tetrahedra that $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ (and hence a free non-Abelian group) embeds in the group of isometries of $\mathbf{R}^3$. For the general problem of the existence of free subgroups of topological groups see the literature quoted in [28, p. 681].

Part (c) for S^2 was first proved by Sierpiński [36]. Further results on the embedding of free products into SO_3 and $SL_2(\mathbf{R})$ are due to Balcerzyk and Mycielski [2]; see also Nisnewitsch [32]. Dekker [7, 8] made an extensive investigation into higher dimensions and the non-Euclidean cases, proving part (c) for H^2 and S^n (except for the case of S^4) and part (a) for S^n provided $n \equiv -1 \pmod 4$. Part (a) for $\mathbf{R}^n$ was proved by Dekker [9] and, independently, by Mycielski and Świerczkowski [29]. The remaining cases of parts (a) and (c) for groups of rank 2 were proved recently by Deligne and Sullivan [11] (part (a) for S^n, $n \equiv 1 \pmod 4$) and Borel [5] (part (c) for S^4). The positive result in part (b) is a consequence of well-known facts about $PSL_2(\mathbf{R})$ (first pointed out in [5]), while the negative result is a consequence of a theorem on $PSL_2(\mathbf{R})$ due to Siegel [35] (see § 6).

The proof of Theorem 1 to be given below will assume all the aforementioned results about the existence of free groups of rank 2. An important fact is that a free group of rank 2 has a free subgroup of rank $\aleph_0$: if σ, τ freely generate the group, F, of rank 2 then $\{\sigma^i \tau \sigma^{-i} : i = 0, 1, 2, ...\}$ is a set of free generators of a subgroup of F, and the same is true of $\{\sigma^i \tau^i : i = 1, 2, ...\}$. More generally (see [21, p. 195]) a free product $A * B$ must have a free subgroup of rank $\aleph_0$ unless A or B is a one-element group or $A \cong \mathbf{Z}_2 \cong B$.

We shall also consider elliptic spaces, L^n, which are represented by S^n, with antipodal points identified. Hence the isometry group of L^n is SO_{n+1}, if n is even, or $SO_{n+1}/\{\pm I\}$, if n is odd. Note that any fixed-point free or locally commutative free subgroup of $G(S^n)$ induces such a subgroup of L^n's isometry group. If a nontrivial (reduced) word w became the identity or gained a fixed point when viewed as acting on L^n, then w^2 would be the identity or have a fixed point as a member of SO_{n+1}. Furthermore, if two words, u and v, share a fixed point on L^n then u^2 and v^2 share a fixed point on S^n. Since, in a free group, u and v commute if and only if u^2 and v^2 do ([21, p. 41]), this shows that a locally commutative free subgroup of SO_{n+1} induces one of the same rank in L^n's isometry group.

§ 3. A PRELIMINARY THEOREM ABOUT METRIC SPACES

The passage from a free group of rank 2 to one of rank $2^{\aleph_0}$ with the same fixed-point properties utilizes the following general theorem of Mycielski [25].

THEOREM 2. *Let X be a complete, separable metric space with no isolated points and suppose that, for each $i < \infty$, R_i is a nowhere dense subset of some finite product X^{r_i}. Then there is a perfect subset F of X which avoids each R_i in the sense that any r_i-tuple of distinct elements of F does not lie in R_i.*

The proof of this theorem is not difficult: one constructs a tree of open sets such that no sequence from distinct nodes at level m lies in any R_i with $i \leqslant m$. Then, provided the open sets are small enough (precisely, their diameters converge to zero along branches, and the closure of any node is contained in one of the open sets at the previous level), F may be obtained as the collection of points lying in the intersections along infinite branches of the tree.