§3. A Preliminary Theorem About Metric Spaces

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The proof of Theorem 1 to be given below will assume all the aforementioned results about the existence of free groups of rank 2. An important fact is that a free group of rank 2 has a free subgroup of rank $\aleph_0$: if σ, τ freely generate the group, F, of rank 2 then $\{\sigma^i \tau \sigma^{-i} : i = 0, 1, 2, ...\}$ is a set of free generators of a subgroup of F, and the same is true of $\{\sigma^i \tau^i : i = 1, 2, ...\}$. More generally (see [21, p. 195]) a free product $A * B$ must have a free subgroup of rank $\aleph_0$ unless A or B is a one-element group or $A \cong \mathbf{Z}_2 \cong B$.

We shall also consider elliptic spaces, L^n, which are represented by S^n, with antipodal points identified. Hence the isometry group of L^n is SO_{n+1}, if n is even, or $SO_{n+1}/\{\pm I\}$, if n is odd. Note that any fixed-point free or locally commutative free subgroup of $G(S^n)$ induces such a subgroup of L^n's isometry group. If a nontrivial (reduced) word w became the identity or gained a fixed point when viewed as acting on L^n, then w^2 would be the identity or have a fixed point as a member of SO_{n+1}. Furthermore, if two words, u and v, share a fixed point on L^n then u^2 and v^2 share a fixed point on S^n. Since, in a free group, u and v commute if and only if u^2 and v^2 do ([21, p. 41]), this shows that a locally commutative free subgroup of SO_{n+1} induces one of the same rank in L^n's isometry group.

§ 3. A Preliminary Theorem About Metric Spaces

The passage from a free group of rank 2 to one of rank $2^{\aleph_0}$ with the same fixed-point properties utilizes the following general theorem of Mycielski [25].

THEOREM 2. *Let X be a complete, separable metric space with no isolated points and suppose that, for each $i < \infty$, R_i is a nowhere dense subset of some finite product X^{r_i}. Then there is a perfect subset F of X which avoids each R_i in the sense that any r_i-tuple of distinct elements of F does not lie in R_i.*

The proof of this theorem is not difficult: one constructs a tree of open sets such that no sequence from distinct nodes at level m lies in any R_i with $i \leqslant m$. Then, provided the open sets are small enough (precisely, their diameters converge to zero along branches, and the closure of any node is contained in one of the open sets at the previous level), F may be obtained as the collection of points lying in the intersections along infinite branches of the tree.

For various applications and generalizations of this result, see [25, 26]. The separability condition is not essential, but its presence allows the proof to be carried out without using the Axiom of Choice.

Our applications of this theorem will involve finding appropriate collections of algebraic or analytic surfaces R_i, such that the set F that avoids them will be the desired set of free generators. The fact that the desired free group of rank two (and hence $\aleph_0$) exists will be used to verify that each R_i is indeed nowhere dense.

§ 4. SPHERES

First we prove Theorem 1 (a) for $S^n (n \text{ odd}, n \geqslant 3)$. Let $A = \{\sigma \in SO_{n+1} : \sigma$ has a fixed point in $S^n\}$; therefore $\sigma \in A$ if and only if $\det(\sigma - I) = 0$. For each nonidentity reduced group word w in m variables, let $R_w = f_w^{-1}(A)$; thus $(\sigma_1, ..., \sigma_m) \in R_w$ if and only if $w(\sigma_1, ..., \sigma_m)$ has a fixed point. It is enough to show that each R_w is nowhere dense, for then Theorem 2 may be applied to the countable set of relations $\{R_w\}$ to get a perfect set $F \subseteq SO_{n+1}$. Since F avoids each R_w, no word using elements of F has a fixed point on S^n. This implies, in particular, that no such word equals the identity, and so F is the desired set of rotations.

To see that each R_w is nowhere dense, we view $SO(n+1)^m$ as a (connected) analytic submanifold of $\mathbf{R}^{m(n+1)^2}$ (of dimension $\frac{1}{2} n(n+1)m$). We need an analytic $f: SO^m_{n+1} \to \mathbf{R}$ such that $R_w = f^{-1}(\{0\})$. Such a function exists because membership in R_w is equivalent to the condition that $+1$ is an eigenvalue of w. Hence we may simply let

$$f(\sigma_1, ..., \sigma_m) = \det\big(w(\sigma_1, ..., \sigma_m) - I\big).$$

Since f is a polynomial in the $m(n+1)^2$ entries of the σ_i (this uses the fact that $\det(\sigma_i) = 1$ to obtain that each entry of σ_i^{-1} is a polynomial in the entries of σ_i), f is analytic on SO^m_{n+1}.

Since f is continuous, R_w is closed, so it remains to show that R_w's interior is empty. Suppose not. Since SO^m_{n+1} is connected, an analytic function that vanishes on a nonempty open set must vanish everywhere. Hence $R_w = SO^m_{n+1}$, which contradicts the existence of a free subgroup of SO_{n+1} of rank m which is fixed-point free (which was proved in [7, 11]). Alternatively, $R_w = SO^m_{n+1}$ contradicts Theorem 1 of [5] which asserts that