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Kapitel: §4. Spheres

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For various applications and generalizations of this result, see [25, 26]. The separability condition is not essential, but its presence allows the proof to be carried out without using the Axiom of Choice.

Our applications of this theorem will involve finding appropriate collections of algebraic or analytic surfaces R_i, such that the set F that avoids them will be the desired set of free generators. The fact that the desired free group of rank two (and hence $\aleph_0$) exists will be used to verify that each R_i is indeed nowhere dense.

§ 4. Spheres

First we prove Theorem 1 (a) for S^n(n odd, $n \geqslant 3$). Let $A = \{\sigma \in SO_{n+1} : \sigma$ has a fixed point in $S^n\}$; therefore $\sigma \in A$ if and only if $\det(\sigma - I) = 0$. For each nonidentity reduced group word w in m variables, let $R_w = f_w^{-1}(A)$; thus $(\sigma_1, \ldots, \sigma_m) \in R_w$ if and only if $w(\sigma_1, \ldots, \sigma_m)$ has a fixed point. It is enough to show that each R_w is nowhere dense, for then Theorem 2 may be applied to the countable set of relations $\{R_w\}$ to get a perfect set $F \subseteq SO_{n+1}$. Since F avoids each R_w, no word using elements of F has a fixed point on S^n. This implies, in particular, that no such word equals the identity, and so F is the desired set of rotations.

To see that each R_w is nowhere dense, we view $SO(n+1)^m$ as a (connected) analytic submanifold of $\mathbf{R}^{m(n+1)^2}$ (of dimension $\frac{1}{2}n(n+1)m$). We need an analytic $f: SO_{n+1}^m \to \mathbf{R}$ such that $R_w = f^{-1}(\{0\})$. Such a function exists because membership in R_w is equivalent to the condition that $+1$ is an eigenvalue of w. Hence we may simply let

$$f(\sigma_1, \ldots, \sigma_m) = \det\big(w(\sigma_1, \ldots, \sigma_m) - I\big).$$

Since f is a polynomial in the $m(n+1)^2$ entries of the σ_i (this uses the fact that $\det(\sigma_i) = 1$ to obtain that each entry of σ_i^{-1} is a polynomial in the entries of σ_i), f is analytic on SO_{n+1}^m.

Since f is continuous, R_w is closed, so it remains to show that R_w's interior is empty. Suppose not. Since SO_{n+1}^m is connected, an analytic function that vanishes on a nonempty open set must vanish everywhere. Hence $R_w = SO_{n+1}^m$, which contradicts the existence of a free subgroup of SO_{n+1} of rank m which is fixed-point free (which was proved in [7, 11]). Alternatively, $R_w = SO_{n+1}^m$ contradicts Theorem 1 of [5] which asserts that

$f_w(SO^m_{n+1})$ is not contained in a proper algebraic subset (in this case, A) of SO_{n+1}. This completes the proof of Theorem 1 (a) for S^n.

Next, consider Theorem 1 (c) for S^n. First observe that this can be proved for SO_3 by the technique above, if A is taken to consist simply of the identity. This is because the action of SO_3 on S^2 is locally commutative, so all that is needed is a perfect set of free generators, which in turn requires only that each R_w be nowhere dense. Theorem 1 of [5] again applies, because A is an algebraic set: membership in A is equivalent to the simultaneous vanishing of $(n+1)^2$ polynomials which, by using a sum of squares, is equivalent to the vanishing of a single polynomial. For higher dimensions, we appeal to the technique used by Borel to get locally commutative free subgroups of SO_{n+1}. In [5, p. 162] he showed that, if $n \geqslant 2$, SO_3 may be represented as a subgroup H of SO_{n+1} where H's action on S^n is locally commutative. Hence the perfect free generating set in SO_3 yields a perfect subset of H which is the desired free generating set in SO_{n+1}.

§ 5. Euclidean Spaces

For the Euclidean case of Theorem 1, it suffices to consider $\mathbf{R}^3$, since any isometry of $\mathbf{R}^3$ can be extended to one in higher dimensions by simply fixing the additional coordinates; this introduces no new fixed points. Now, $\mathbf{R}^3$ can be handled in a way entirely similar to S^n. Any orientation-preserving isometry of $\mathbf{R}^3$ is a screw-motion, i.e. a rotation $\rho \in SO_3$ followed by a translation τ. Such isometries may be represented as elements of $SL_4(\mathbf{R})$ as follows: if $\sigma = \tau\rho$ where ρ corresponds to $(a_{ij}) \in SO_3$ and τ is a translation by (v_1, v_2, v_3), then identify σ with the matrix

$$\begin{pmatrix} & & & v_1 \\ & a_{ij} & & v_2 \\ & & & v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since composition of isometries corresponds to matrix multiplication, this shows that $G(\mathbf{R}^3)$ may be viewed as a connected (6-dimensional) analytic submanifold of $\mathbf{R}^{12}$. Now, the proof can proceed exactly as for spheres, once it is shown that the existence of a fixed point is equivalent to the