# §8. A Paradoxical Decomposition Using Borel Sets

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orems 1 (c) and 4 (c) yield a subset that is both a third of $H^2$ and a $2^{\aleph_0}$'th part of $H^2$.

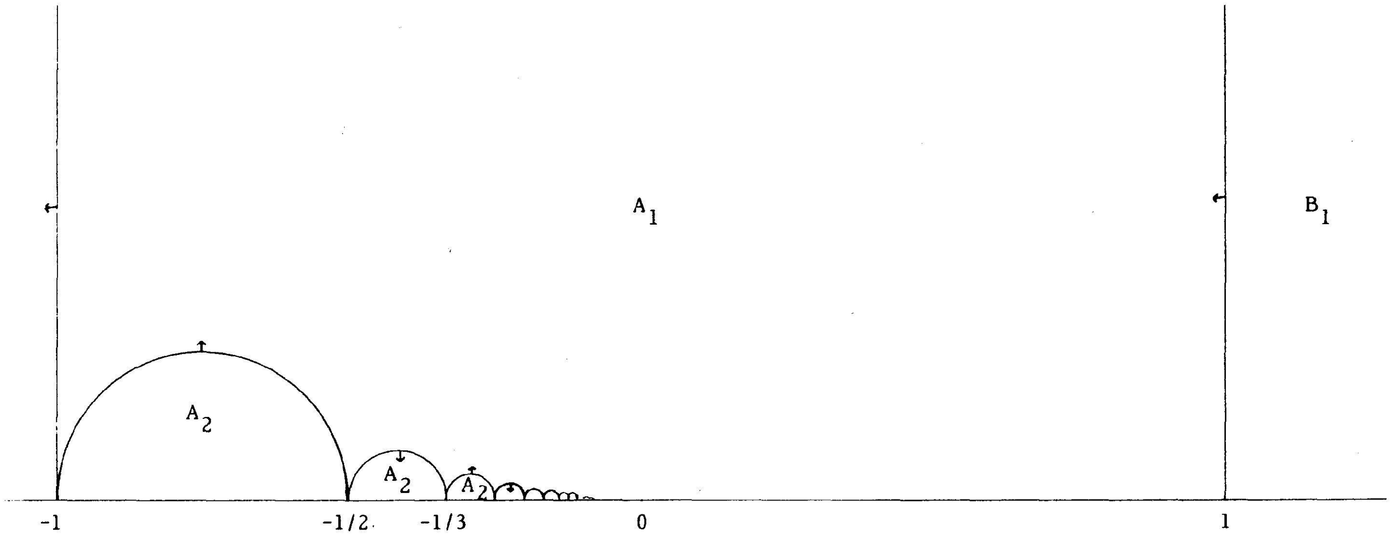
## § 8. A PARADOXICAL DECOMPOSITION USING BOREL SETS

THEOREM 8. *If $n \geqslant 2$, then any system of countably many congruences involving countably many sets (as in Theorem 6) is satisfiable using a partition of $H^n$ into Borel sets and isometries.*

*Proof.* Consider $H^2$ first, and let $F$ be a free subgroup of $PSL_2(\mathbf{Z})$ whose rank equals the number of congruences to be satisfied; $F$ may be obtained as a subgroup of the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose. Theorem 6 is proved by first constructing, by induction, a partition of $F$ that satisfies the given system using left multiplication in $F$. Then it is easy to transfer this decomposition to a set on which $F$'s action is fixed-point free by using a choice set for the $F$-orbits. In general, this requires the Axiom of Choice, and yields nonmeasurable sets. But, because $F$ is a discrete subgroup of $PSL_2(\mathbf{R})$, there is a fundamental region for $F$'s action on $H^2$. In fact (see [18]) there is a (hyperbolic) polygon such that no two points of the polygon's interior lie in the same $F$-orbit, and all points in $H^2$ are in the $F$-orbit of some point in the closure of the polygon. The boundary of this polygon consists of a countable number of sides (open hyperbolic segments) and vertices. Since $F$ maps vertices to vertices and sides to sides, there is a choice set $M$ for the $F$-orbits that consists of the interior of the polygon together with some of the vertices and some of the sides. Clearly, $M$ is a Borel set. Now, if $B_n$ is one of the sets of the partition of $F$, then let $A_n = \cup\{\sigma(M) : \sigma \in B_n\}$. This yields a partition of $H^2$ into Borel sets $A_n$ which satisfy the given congruences. The result for higher dimensions follows by simple using the standard projection of $H^n$ onto $H^2$ to define the pieces of a partition of $H^n$.

COROLLARY. *If $n \geqslant 2$ then $H^n$ is paradoxical using Borel sets. In fact, there are pairwise disjoint Borel sets, $A_1, A_2, B_1, B_2$ and isometries $\sigma_1, \sigma_2, \tau_1, \tau_2 \in G(H^n)$ such that $H^n = \sigma_1(A_1) \cup \sigma_2(A_2) = \tau_1(B_1) \cup \tau_2(B_2)$. Moreover, there is a Borel set $E$ which is simultaneously a half, a third, ..., an $\aleph_0$'th part of $H^2$.*

This corollary shows that the subsets of $H^n$ provided by parts (b) of (c) of Theorem 4 can be taken to be Borel sets in the case $\kappa = \aleph_0$. This result is completely constructive. For instance, if one labels the quadrilaterals of the tesselation corresponding to the discrete free group generated by $\sigma$ and $\tau$ (where $\sigma(z) = \dfrac{z}{2z+1}$ and $\tau(z) = z + 2$) and then transfers the paradoxical decomposition of a free group of rank two to $H^2$ via the labelled quadrilaterals, one obtains the partition of $H^2$ into four sets $A_1, A_2, B_1$ and $B_2$ illustrated in the figure below. Since $H^2 = A_1 \cup \sigma(A_2) = B_1 \cup \tau(B_2)$, this yields an explicit paradoxical decomposition of the hyperbolic plane using very simple sets. For another pictorially simple paradox in $H^2$ see [41, Fig. 5.2].




These results are completely opposite to the situation in $S^2$ and $\mathbf{R}^n$. Because of surface Lebesgue measure on $S^n$, it is obvious that parts (b) and (c) of Theorem 4 cannot be witnessed by measurable sets. For example, if $m$ denotes surface Lebesgue measure and $E$, a measurable set, is a $\lambda$'th part of $S^n$, then $m(E) = \dfrac{1}{\lambda}$, if $\lambda$ is finite, and $m(E) = 0$ if $\lambda$ is infinite. The case of $\mathbf{R}^n$ is subtler because $\mathbf{R}^n$ has infinite measure; the following result of Mycielski [27] is relevant.

THEOREM 9. *There is a finitely additive measure* $\mu$ *on the collection of Lebesgue measurable subsets of* $\mathbf{R}^n$ *which is invariant under all similarities and satisfies* $\mu(\mathbf{R}^n) = 1$.

Because the similarity groups in $\mathbf{R}^1$ and $\mathbf{R}^2$ are solvable, the theorem of Banach mentioned in § 7 shows that, in these two cases, the measure can be taken to be defined on all sets.

Note that for $\kappa$ uncountable parts (b) and (c) of Theorem 4 cannot be witnessed by Borel subsets of $H^n$. Suppose, for example, that $\kappa$ is uncountable and the sets of Theorem 4(b) are all Borel. Since Borel sets have the Property of Baire, each $A_\alpha$ may be written as $R_\alpha \Delta M_\alpha$ where $R_\alpha$ is open and $M_\alpha$ is meager. But each $A_\alpha$, being Borel equidecomposable to all of $H^2$, is nonmeager, whence each $R_\alpha$ is nonempty. It follows that the $R_\alpha$ are pairwise disjoint, which contradicts the separability of $H^2$. A similar argument shows that the sets cannot all be Lebesgue measurable either.

Let us point out how the proof of Theorem 9 breaks down in hyperbolic space. Theorem 9 is based on the fact that $\mathbf{R}^n$ is a union of countably many sets $B_r$ of finite Lebesgue measure satisfying: for any isometry $\sigma$, $m(B_r \Delta \sigma(B_r))/m(B_r) \to 0$ as $r \to \infty$. Simply let $B_r$ be the ball of radius $r$ centered at the origin. Because Theorem 9 is false for $H^n$ if $n \geqslant 2$, there can be no such sequence of almost invariant sets of finite (hyperbolic) measure in $H^n$.

## § 9. Linear Transformations of the Euclidean Plane

Paradoxical decompositions in the plane are possible if one allows the use of area-preserving affine transformations. This was first realized by von Neumann [31], who showed that a square is paradoxical using this expansion of the isometry group. In fact, it is sufficient to consider the group generated by $SL_2(\mathbf{Z})$ and all translations; see [39]. In this section we discuss how the results of this paper are affected by considering linear, or affine, transformations instead of just isometries.

Let us consider the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$. The two matrices, $\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ and $\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$ freely generate a subgroup of $SL_2(\mathbf{Z})$, no nonidentity element of which has a fixed point in $\mathbf{R}^2\backslash\{0\}$; this follows from the result of Magnus and Neumann mentioned in § 6, since an element of $SL_2(\mathbf{Z})$ has a nonzero fixed point in $\mathbf{R}^2$ if and only if it has trace 2. It follows by the technique of § 4 that $SL_2(\mathbf{R})$ has a free subgroup with a perfect set of free generators whose action on $\mathbf{R}^2\backslash\{0\}$ is fixed-point free. Therefore the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$ satisfies all the conclusions of Theorems 4 and 6.

Using techniques of functional analysis, J. Rosenblatt and R. Kallman (unpublished) have recently shown that the Lebesgue measurable subsets of $\mathbf{R}^n\backslash\{0\}$ ($n \geqslant 2$) do not bear a finitely additive, $SL_n(\mathbf{Z})$-invariant measure of total measure one. (For $n \geqslant 3$ this uses the fact that $SL_n(\mathbf{Z})$ has Kazhdan's Property T, while the $\mathbf{R}^2$ case uses specific facts about representations of