

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 30 (1984)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LARGE FREE GROUPS OF ISOMETRIES AND THEIR GEOMETRICAL USES
Autor: Mycielski, Jan / Wagon, Stan
Kapitel: §8. A Paradoxical Decomposition Using Borel Sets
DOI: <https://doi.org/10.5169/seals-53829>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 03.07.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

orems 1 (c) and 4 (c) yield a subset that is both a third of H^2 and a 2^{\aleph_0} 'th part of H^2 .

§ 8. A PARADOXICAL DECOMPOSITION USING BOREL SETS

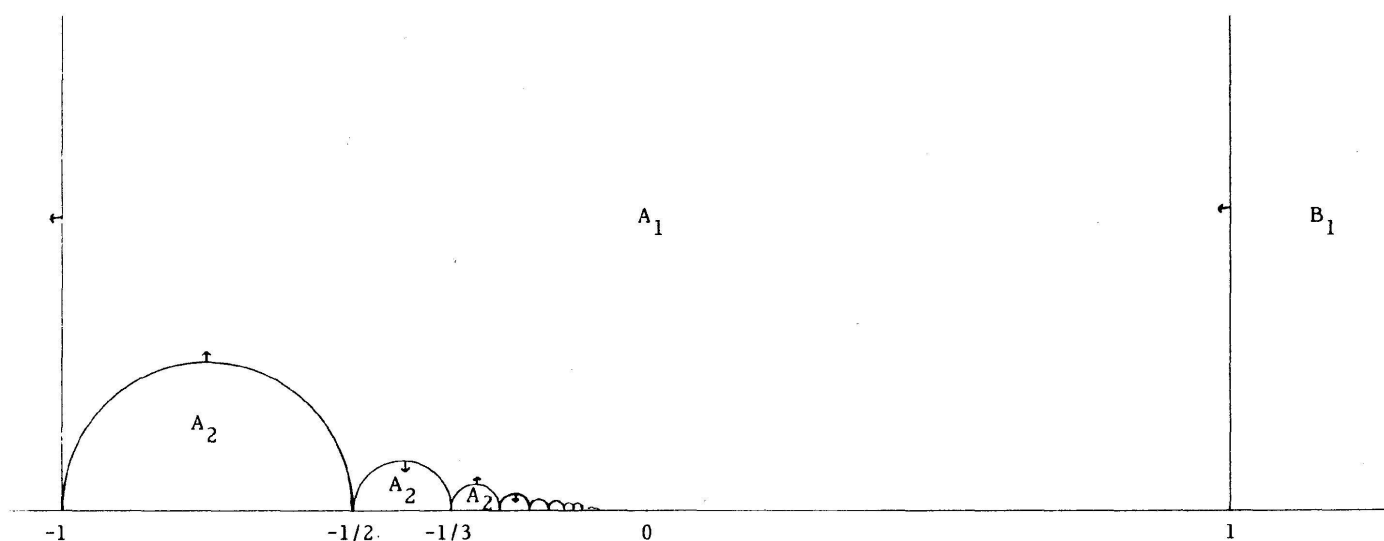
THEOREM 8. *If $n \geq 2$, then any system of countably many congruences involving countably many sets (as in Theorem 6) is satisfiable using a partition of H^n into Borel sets and isometries.*

Proof. Consider H^2 first, and let F be a free subgroup of $PSL_2(\mathbf{Z})$ whose rank equals the number of congruences to be satisfied; F may be obtained as a subgroup of the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose. Theorem 6 is proved by first constructing, by induction, a partition of F that satisfies the given system using left multiplication in F . Then it is easy to transfer this decomposition to a set on which F 's action is fixed-point free by using a choice set for the F -orbits. In general, this requires the Axiom of Choice, and yields nonmeasurable sets. But, because F is a discrete subgroup of $PSL_2(\mathbf{R})$, there is a fundamental region for F 's action on H^2 . In fact (see [18]) there is a (hyperbolic) polygon such that no two points of the polygon's interior lie in the same F -orbit, and all points in H^2 are in the F -orbit of some point in the closure of the polygon. The boundary of this polygon consists of a countable number of sides (open hyperbolic segments) and vertices. Since F maps vertices to vertices and sides to sides, there is a choice set M for the F -orbits that consists of the interior of the polygon together with some of the vertices and some of the sides. Clearly, M is a Borel set. Now, if B_n is one of the sets of the partition of F , then let $A_n = \cup \{\sigma(M) : \sigma \in B_n\}$. This yields a partition of H^2 into Borel sets A_n which satisfy the given congruences. The result for higher dimensions follows by simply using the standard projection of H^n onto H^2 to define the pieces of a partition of H^n .

COROLLARY. *If $n \geq 2$ then H^n is paradoxical using Borel sets. In fact, there are pairwise disjoint Borel sets, A_1, A_2, B_1, B_2 and isometries $\sigma_1, \sigma_2, \tau_1, \tau_2 \in G(H^n)$ such that $H^n = \sigma_1(A_1) \cup \sigma_2(A_2) = \tau_1(B_1) \cup \tau_2(B_2)$. Moreover, there is a Borel set E which is simultaneously a half, a third, ..., an \aleph_0 'th part of H^2 .*

This corollary shows that the subsets of H^n provided by parts (b) of (c) of Theorem 4 can be taken to be Borel sets in the case $\kappa = \aleph_0$. This

result is completely constructive. For instance, if one labels the quadrilaterals of the tessellation corresponding to the discrete free group generated by σ and τ (where $\sigma(z) = \frac{z}{2z+1}$ and $\tau(z) = z+2$) and then transfers the paradoxical decomposition of a free group of rank two to H^2 via the labelled quadrilaterals, one obtains the partition of H^2 into four sets A_1, A_2, B_1 and B_2 illustrated in the figure below. Since $H^2 = A_1 \cup \sigma(A_2) = B_1 \cup \tau(B_2)$, this yields an explicit paradoxical decomposition of the hyperbolic plane using very simple sets. For another pictorially simple paradox in H^2 see [41, Fig. 5.2].



These results are completely opposite to the situation in S^2 and \mathbf{R}^n . Because of surface Lebesgue measure on S^n , it is obvious that parts (b) and (c) of Theorem 4 cannot be witnessed by measurable sets. For example, if m denotes surface Lebesgue measure and E , a measurable set, is a λ 'th part of S^n , then $m(E) = \frac{1}{\lambda}$, if λ is finite, and $m(E) = 0$ if λ is infinite. The case of \mathbf{R}^n is subtler because \mathbf{R}^n has infinite measure; the following result of Mycielski [27] is relevant.

THEOREM 9. *There is a finitely additive measure μ on the collection of Lebesgue measurable subsets of \mathbf{R}^n which is invariant under all similarities and satisfies $\mu(\mathbf{R}^n) = 1$.*

Because the similarity groups in \mathbf{R}^1 and \mathbf{R}^2 are solvable, the theorem of Banach mentioned in § 7 shows that, in these two cases, the measure can be taken to be defined on all sets.

Note that for κ uncountable parts (b) and (c) of Theorem 4 cannot be witnessed by Borel subsets of H^n . Suppose, for example, that κ is uncountable

and the sets of Theorem 4 (b) are all Borel. Since Borel sets have the Property of Baire, each A_α may be written as $R_\alpha \Delta M_\alpha$ where R_α is open and M_α is meager. But each A_α , being Borel equidecomposable to all of H^2 , is nonmeager, whence each R_α is nonempty. It follows that the R_α are pairwise disjoint, which contradicts the separability of H^2 . A similar argument shows that the sets cannot all be Lebesgue measurable either.

Let us point out how the proof of Theorem 9 breaks down in hyperbolic space. Theorem 9 is based on the fact that \mathbf{R}^n is a union of countably many sets B_r of finite Lebesgue measure satisfying: for any isometry σ , $m(B_r \Delta \sigma(B_r))/m(B_r) \rightarrow 0$ as $r \rightarrow \infty$. Simply let B_r be the ball of radius r centered at the origin. Because Theorem 9 is false for H^n if $n \geq 2$, there can be no such sequence of almost invariant sets of finite (hyperbolic) measure in H^n .

§ 9. LINEAR TRANSFORMATIONS OF THE EUCLIDEAN PLANE

Paradoxical decompositions in the plane are possible if one allows the use of area-preserving affine transformations. This was first realized by von Neumann [31], who showed that a square is paradoxical using this expansion of the isometry group. In fact, it is sufficient to consider the group generated by $SL_2(\mathbf{Z})$ and all translations; see [39]. In this section we discuss how the results of this paper are affected by considering linear, or affine, transformations instead of just isometries.

Let us consider the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2 \setminus \{0\}$. The two matrices, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ freely generate a subgroup of $SL_2(\mathbf{Z})$, no nonidentity element of which has a fixed point in $\mathbf{R}^2 \setminus \{0\}$; this follows from the result of Magnus and Neumann mentioned in § 6, since an element of $SL_2(\mathbf{Z})$ has a nonzero fixed point in \mathbf{R}^2 if and only if it has trace 2. It follows by the technique of § 4 that $SL_2(\mathbf{R})$ has a free subgroup with a perfect set of free generators whose action on $\mathbf{R}^2 \setminus \{0\}$ is fixed-point free. Therefore the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2 \setminus \{0\}$ satisfies all the conclusions of Theorems 4 and 6.

Using techniques of functional analysis, J. Rosenblatt and R. Kallman (unpublished) have recently shown that the Lebesgue measurable subsets of $\mathbf{R}^n \setminus \{0\}$ ($n \geq 2$) do not bear a finitely additive, $SL_n(\mathbf{Z})$ -invariant measure of total measure one. (For $n \geq 3$ this uses the fact that $SL_n(\mathbf{Z})$ has Kazhdan's Property T, while the \mathbf{R}^2 case uses specific facts about representations of