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orems 1 (c) and 4 (c) yield a subset that is both a third of H^2 and a $2^{\aleph_0}$'th part of H^2.

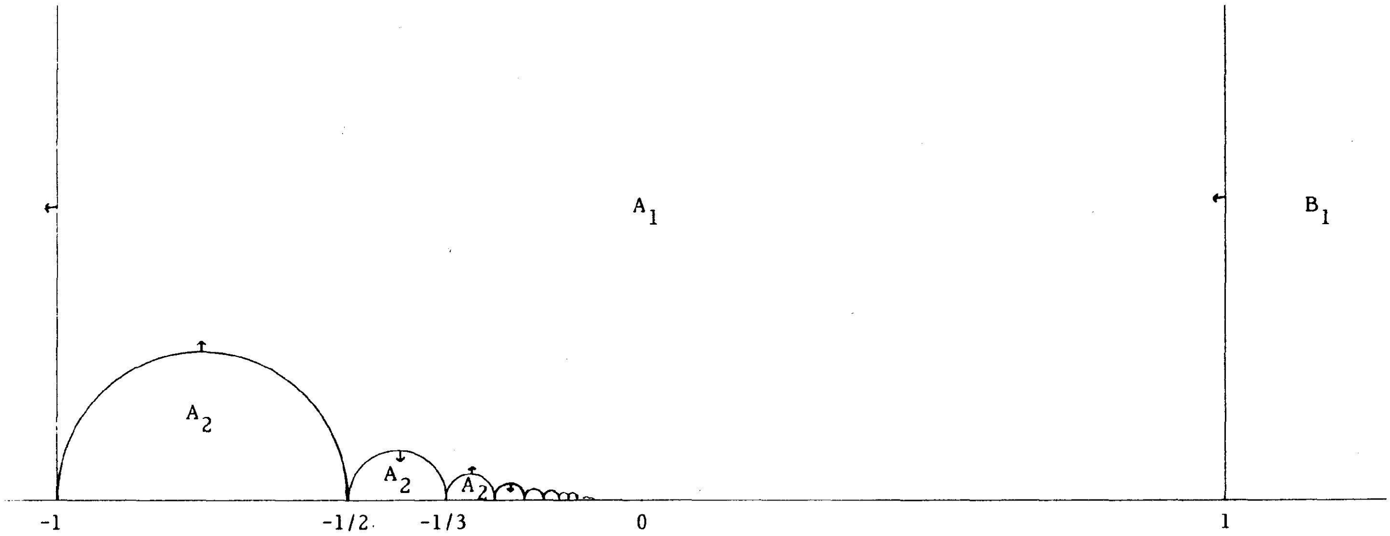
§ 8. A Paradoxical Decomposition Using Borel Sets

THEOREM 8. *If* $n \geqslant 2$, *then any system of countably many congruences involving countably many sets (as in Theorem 6) is satisfiable using a partition of* H^n *into Borel sets and isometries.*

Proof. Consider H^2 first, and let F be a free subgroup of $PSL_2(\mathbf{Z})$ whose rank equals the number of congruences to be satisfied; F may be obtained as a subgroup of the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose. Theorem 6 is proved by first constructing, by induction, a partition of F that satisfies the given system using left multiplication in F. Then it is easy to transfer this decomposition to a set on which F's action is fixed-point free by using a choice set for the F-orbits. In general, this requires the Axiom of Choice, and yields nonmeasurable sets. But, because F is a discrete subgroup of $PSL_2(\mathbf{R})$, there is a fundamental region for F's action on H^2. In fact (see [18]) there is a (hyperbolic) polygon such that no two points of the polygon's interior lie in the same F-orbit, and all points in H^2 are in the F-orbit of some point in the closure of the polygon. The boundary of this polygon consists of a countable number of sides (open hyperbolic segments) and vertices. Since F maps vertices to vertices and sides to sides, there is a choice set M for the F-orbits that consists of the interior of the polygon together with some of the vertices and some of the sides. Clearly, M is a Borel set. Now, if B_n is one of the sets of the partition of F, then let $A_n = \cup\{\sigma(M) : \sigma \in B_n\}$. This yields a partition of H^2 into Borel sets A_n which satisfy the given congruences. The result for higher dimensions follows by simple using the standard projection of H^n onto H^2 to define the pieces of a partition of H^n.

COROLLARY. *If* $n \geqslant 2$ *then* H^n *is paradoxical using Borel sets. In fact, there are pairwise disjoint Borel sets,* A_1, A_2, B_1, B_2 *and isometries* $\sigma_1, \sigma_2, \tau_1, \tau_2 \in G(H^n)$ *such that* $H^n = \sigma_1(A_1) \cup \sigma_2(A_2) = \tau_1(B_1) \cup \tau_2(B_2)$. *Moreover, there is a Borel set* E *which is simultaneously a half, a third, ..., an* $\aleph_0$*'th part of* H^2.

This corollary shows that the subsets of H^n provided by parts (b) of (c) of Theorem 4 can be taken to be Borel sets in the case $\kappa = \aleph_0$. This result is completely constructive. For instance, if one labels the quadrilaterals of the tesselation corresponding to the discrete free group generated by σ and τ (where $\sigma(z) = \frac{z}{2z+1}$ and $\tau(z) = z + 2$) and then transfers the paradoxical decomposition of a free group of rank two to H^2 via the labelled quadrilaterals, one obtains the partition of H^2 into four sets A_1, A_2, B_1 and B_2 illustrated in the figure below. Since $H^2 = A_1 \cup \sigma(A_2) = B_1 \cup \tau(B_2)$, this yields an explicit paradoxical decomposition of the hyperbolic plane using very simple sets. For another pictorially simple paradox in H^2 see [41, Fig. 5.2].



These results are completely opposite to the situation in S^2 and $\mathbf{R}^n$. Because of surface Lebesgue measure on S^n, it is obvious that parts (b) and (c) of Theorem 4 cannot be witnessed by measurable sets. For example, if m denotes surface Lebesgue measure and E, a measurable set, is a λ'th part of S^n, then $m(E) = \frac{1}{\lambda}$, if λ is finite, and $m(E) = 0$ if λ is infinite. The case of $\mathbf{R}^n$ is subtler because $\mathbf{R}^n$ has infinite measure; the following result of Mycielski [27] is relevant.

THEOREM 9. *There is a finitely additive measure* μ *on the collection of Lebesgue measurable subsets of* $\mathbf{R}^n$ *which is invariant under all similarities and satisfies* $\mu(\mathbf{R}^n) = 1$.

Because the similarity groups in $\mathbf{R}^1$ and $\mathbf{R}^2$ are solvable, the theorem of Banach mentioned in § 7 shows that, in these two cases, the measure can be taken to be defined on all sets.

Note that for κ uncountable parts (b) and (c) of Theorem 4 cannot be witnessed by Borel subsets of H^n. Suppose, for example, that κ is uncountable and the sets of Theorem 4 (b) are all Borel. Since Borel sets have the Property of Baire, each A_α may be written as $R_\alpha \Delta M_\alpha$ where R_α is open and M_α is meager. But each A_α, being Borel equidecomposable to all of H^2, is nonmeager, whence each R_α is nonempty. It follows that the R_α are pairwise disjoint, which contradicts the separability of H^2. A similar argument shows that the sets cannot all be Lebesgue measurable either.

Let us point out how the proof of Theorem 9 breaks down in hyperbolic space. Theorem 9 is based on the fact that $\mathbf{R}^n$ is a union of countably many sets B_r of finite Lebesgue measure satisfying: for any isometry σ, $m(B_r \Delta \sigma(B_r))/m(B_r) \to 0$ as $r \to \infty$. Simply let B_r be the ball of radius r centered at the origin. Because Theorem 9 is false for H^n if $n \geqslant 2$, there can be no such sequence of almost invariant sets of finite (hyperbolic) measure in H^n.

§ 9. Linear Transformations of the Euclidean Plane

Paradoxical decompositions in the plane are possible if one allows the use of area-preserving affine transformations. This was first realized by von Neumann [31], who showed that a square is paradoxical using this expansion of the isometry group. In fact, it is sufficient to consider the group generated by $SL_2(\mathbf{Z})$ and all translations; see [39]. In this section we discuss how the results of this paper are affected by considering linear, or affine, transformations instead of just isometries.

Let us consider the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$. The two matrices, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ freely generate a subgroup of $SL_2(\mathbf{Z})$, no nonidentity element of which has a fixed point in $\mathbf{R}^2\backslash\{0\}$; this follows from the result of Magnus and Neumann mentioned in § 6, since an element of $SL_2(\mathbf{Z})$ has a nonzero fixed point in $\mathbf{R}^2$ if and only if it has trace 2. It follows by the technique of § 4 that $SL_2(\mathbf{R})$ has a free subgroup with a perfect set of free generators whose action on $\mathbf{R}^2\backslash\{0\}$ is fixed-point free. Therefore the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$ satisfies all the conclusions of Theorems 4 and 6.

Using techniques of functional analysis, J. Rosenblatt and R. Kallman (unpublished) have recently shown that the Lebesgue measurable subsets of $\mathbf{R}^n\backslash\{0\}$ ($n \geqslant 2$) do not bear a finitely additive, $SL_n(\mathbf{Z})$-invariant measure of total measure one. (For $n \geqslant 3$ this uses the fact that $SL_n(\mathbf{Z})$ has Kazhdan's Property T, while the $\mathbf{R}^2$ case uses specific facts about representations of