§9. Linear Transformations of the Euclidean Plane

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and the sets of Theorem 4 (b) are all Borel. Since Borel sets have the Property of Baire, each A_α may be written as $R_\alpha \Delta M_\alpha$ where R_α is open and M_α is meager. But each A_α, being Borel equidecomposable to all of H^2, is nonmeager, whence each R_α is nonempty. It follows that the R_α are pairwise disjoint, which contradicts the separability of H^2. A similar argument shows that the sets cannot all be Lebesgue measurable either.

Let us point out how the proof of Theorem 9 breaks down in hyperbolic space. Theorem 9 is based on the fact that $\mathbf{R}^n$ is a union of countably many sets B_r of finite Lebesgue measure satisfying: for any isometry σ, $m(B_r \Delta \sigma(B_r))/m(B_r) \to 0$ as $r \to \infty$. Simply let B_r be the ball of radius r centered at the origin. Because Theorem 9 is false for H^n if $n \geqslant 2$, there can be no such sequence of almost invariant sets of finite (hyperbolic) measure in H^n.

§ 9. Linear Transformations of the Euclidean Plane

Paradoxical decompositions in the plane are possible if one allows the use of area-preserving affine transformations. This was first realized by von Neumann [31], who showed that a square is paradoxical using this expansion of the isometry group. In fact, it is sufficient to consider the group generated by $SL_2(\mathbf{Z})$ and all translations; see [39]. In this section we discuss how the results of this paper are affected by considering linear, or affine, transformations instead of just isometries.

Let us consider the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$. The two matrices, $\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ and $\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$ freely generate a subgroup of $SL_2(\mathbf{Z})$, no nonidentity element of which has a fixed point in $\mathbf{R}^2\backslash\{0\}$; this follows from the result of Magnus and Neumann mentioned in § 6, since an element of $SL_2(\mathbf{Z})$ has a nonzero fixed point in $\mathbf{R}^2$ if and only if it has trace 2. It follows by the technique of § 4 that $SL_2(\mathbf{R})$ has a free subgroup with a perfect set of free generators whose action on $\mathbf{R}^2\backslash\{0\}$ is fixed-point free. Therefore the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$ satisfies all the conclusions of Theorems 4 and 6.

Using techniques of functional analysis, J. Rosenblatt and R. Kallman (unpublished) have recently shown that the Lebesgue measurable subsets of $\mathbf{R}^n\backslash\{0\}$ ($n \geqslant 2$) do not bear a finitely additive, $SL_n(\mathbf{Z})$-invariant measure of total measure one. (For $n \geqslant 3$ this uses the fact that $SL_n(\mathbf{Z})$ has Kazhdan's Property T, while the $\mathbf{R}^2$ case uses specific facts about representations of

$SL_2(\mathbf{Z})$; see [41; Theorem 11.17].) Thus Theorem 9 does not extend to area-preserving affine transformations. It would be interesting if a paradoxical decomposition of $\mathbf{R}^2 \setminus \{0\}$ using measurable sets, similar to the one illustrated in § 8, could be explicitly constructed. Some sort of paradoxical decomposition using measurable pieces must exist, by a general theorem of Tarski (see [41]), but it is not known if one using just four pieces exists. On the other hand, Belley and Prasad [4] have shown that there is a finitely additive measure on a certain (not too small) Boolean algebra of Borel subsets of $\mathbf{R}^n$ that has total measure one and is invariant under all nonsingular affine transformations of $\mathbf{R}^n$ (not just the measure-preserving ones).

Finally, we mention some unsolved problems about the existence of nice free groups of affine, area-preserving transformations, positive solutions to which would yield (via Theorems 4-6) paradoxical decompositions of $\mathbf{R}^n$. Let $A_n(\mathbf{R})$ denote the group of affine transformations of $\mathbf{R}^n$, i.e., transformations of the form TL, where T is a translation and $L \in GL_n(\mathbf{R})$. Let $SA_n(\mathbf{R})$ be the subgroup obtained by restricting L to $SL_n(\mathbf{R})$, and let $SA_n(\mathbf{Z})$ consist of those TL where $L \in SL_n(\mathbf{Z})$ and T is a translation by a vector in $\mathbf{Z}^n$. Note that $SA_n(\mathbf{Z})$ acts on $\mathbf{Z}^n$. Since $G(\mathbf{R}^3) \subseteq SA_3(\mathbf{R})$, Theorem 1 yields that $SA_3(\mathbf{R})$ has a free non-Abelian subgroup whose action on $\mathbf{R}^3$ is fixed-point free. Consideration of $\mathbf{Z}^3$ instead of $\mathbf{R}^3$ leads to problem 1 below. Problem 2 is an attempt to get a version of these results for $\mathbf{R}^2$ (rather than $\mathbf{R}^2 \setminus \{0\}$, which is treated at the beginning of this section). Only local commutativity is sought because of part (b) of the proposition below. Since $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose freely generate a group of rank two, so do the two transformations:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence perhaps the subgroup of $SA_2(\mathbf{Z})$ which these two transformations generate solves Problem 2 affirmatively. But we are unable to show that this subgroup is locally commutative.

Problems.

1. Does $SA_3(\mathbf{Z})$ have a free subgroup of rank two which is fixed-point free on $\mathbf{Z}^3$?

2. Does $SA_2(\mathbf{R})$ (or $SA_2(\mathbf{Z})$) have a subgroup of rank two which is locally commutative in its action on $\mathbf{R}^2$ (or on $\mathbf{Z}^2$)?

PROPOSITION 10.

(a) *If* $TL \in A_n(\mathbf{R})$ *and* TL *has no fixed points in* $\mathbf{R}^n$, *then* L *has* $+1$ *as an eigenvalue, i.e.,* L *has a fixed point in* $\mathbf{R}^n \setminus \{0\}$.

(b) *If* G *is a subgroup of* $SA_2(\mathbf{R})$ *which is fixed-point free on* $\mathbf{R}^2$ *then* G *is solvable.*

Proof.

(a) Suppose T is a translation by the vector v. Since $L(x) + v = x$ has no solution, the same is true of $(L - I)(x) = -v$, and therefore $\det(L - I) = 0$, i.e., 1 is an eigenvalue of L.

(b) Let $\sigma = TL$ and $\tau = T'L'$ be in G. Then $\sigma\tau = T''LL'$ so part (a) yields that each of L, L', LL' has 1 as an eigenvalue. Since these are 2×2 matrices with determinant 1, this implies that all have trace 2. Hence, choosing an appropriate basis, we have $L = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $L' = \begin{pmatrix} \alpha & \beta \\ \gamma & 2-\alpha \end{pmatrix}$. Then $LL' = \begin{pmatrix} \alpha + b\gamma & * \\ * & 2-\alpha \end{pmatrix}$, and the trace of the latter being 2 yields that $b\gamma = 0$. But if either b or γ equal zero, then L and L' commute, which implies that the commutators $\sigma\tau\sigma^{-1}\tau^{-1}$ and $\sigma^{-1}\tau^{-1}\sigma\tau$ are pure translations. Hence [[G, G], [G, G]] is the identity subgroup, i.e., G is solvable.

Part (b) of the Proposition shows why there is no fixed-point free, non-Abelian free subgroup of $SA_2(\mathbf{R})$. But the following problem is unsolved.

Problem 3. Does there exist a free non-Abelian semigroup in $SA_2(\mathbf{R})$ (or $SA_2(\mathbf{Z})$) whose action on $\mathbf{R}^2$ is fixed-point free?

Part (a) of Proposition 10 brings to light a distinction between the groups $G(\mathbf{R}^n)$ according as n is even or odd. The proof of Theorem 1 for $\mathbf{R}^3$ (§ 5) is essentially the same as the proof for S^{2n+1} given in § 4. Precisely, it is shown that $A = \{\sigma \in G(\mathbf{R}^3) : \sigma$ has a fixed point in $\mathbf{R}^3\}$ is nowhere dense and, in fact, each $R_w = f_w^{-1}(A)$ is nowhere dense in the appropriate product, where w is any group word in finitely many variables. While this is sufficient to get the existence of perfect free generating sets of fixed-point free subgroups in $\mathbf{R}^3$ and beyond, the set A can fail to be nowhere dense in the higher dimensions. Indeed, consider $\mathbf{R}^{2n}$, $n \geqslant 1$. Letting $\pi : G(\mathbf{R}^{2n}) \to SO_{2n}$ be the canonical homomorphism, it follows from part (a) of Proposition 10 that $G(\mathbf{R}^n) \setminus A \subseteq \pi^{-1}(B)$, where $B = \{L \in SO_{2n} : L$ has 1 as an eigenvalue$\}$. It is easy to see that B is nowhere dense and it follows that the same is true of $\pi^{-1}(B)$; i.e., A has a nowhere dense complement. In odd dimensions, however, the situation in $\mathbf{R}^3$ is typical, as the following proposition shows.

PROPOSITION 11. *If* $n \geqslant 1$ *is odd then* $A = \{\sigma \in G(\mathbf{R}^n)\colon \sigma$ *has a fixed point in* $\mathbf{R}^n\}$ *is a nowhere dense subset of* $G(\mathbf{R}^n)$.

Proof. It is an easy linear algebra exercise (generalizing Proposition 10 (a) above) to see that $\sigma = TL$ has a fixed point in $\mathbf{R}^n$ if and only if the translation vector of T is orthogonal to all vectors fixed by L. Since there is a basis for the fixed space of L that consists of vectors whose entries are polynomials in the entries of L (Gaussian elimination and scaling), this latter condition on TL is equivalent to the vanishing of a polynomial in the entries of σ. But the condition is not universally true in $G(\mathbf{R}^n)$ since any pure translation has no fixed points; therefore the technique introduced in § 4 implies that A is nowhere dense, as desired.

This proposition, in exactly the same cases, is valid for SO_{n+1}'s action on S^n (see § 4). The following extension of these results is a refinement of the theorems on the existence of free, fixed-point free groups of isometries of rank m: it shows that in these cases almost all (from the category point of view) m-tuples of isometries are free generators of fixed-point free groups of isometries.

PROPOSITION 12. *Suppose* n *is odd and* $n \geqslant 3$, *and* X *is one of* $\mathbf{R}^n$ *or* S^n. *Then any* m *elements of* $G(X)$, *with the exception of a meager set in* $G(X)^m$, *are free generators of a fixed-point free subgroup of* $G(X)$.

Proof. For the spherical case this follows from § 4, where it was shown that $\cup\{R_w\colon w$ is a group word in m variables$\}$ is comeager. The Euclidean case is proved by observing (see Proposition 11's proof and § 5) that there is a function p that is a polynomial in the entries of $\sigma_1, \ldots, \sigma_m$ such that $p = 0$ if and only if $f_w(\sigma_1, \ldots, \sigma_m) \in A$. Since, by the rank two case of Theorem 1 (a), f is not identically zero, $f_w^{-1}(A)$ is nowhere dense. Therefore the union over all words in m variables is meager, as desired.