TRIPLY UNIVERSAL ENTIRE FUNCTION

L'Enseignement Mathématique, t. 30 (1984), p. 269-274

A TRIPLY UNIVERSAL ENTIRE FUNCTION

by Charles BLAIR and Lee RUBEL

Dedicated to the Memory of Ernst Straus

It is an interesting and amusing fact that there exist entire functions f of one complex variable that are universal in the sense that by applying simple analytical operations to f, and then taking limits, one can arrive at any entire function whatsoever. Let us consider four possible ways of making this precise:

(a) the set of derivatives $f^{(n)}$, $n = 0, 1, 2, \ldots$ is dense,

(b) the set of positive integer translates f_n, $n = 0, 1, 2, \ldots$ is dense,

(c) a set of iterated integrals $f^{(-n)}$, $n = 0, 1, 2, \ldots$ is dense,

(d) the set of finite linear combinations of iterates $f^{[n]}$, $n = 1, 2, 3, \ldots$ is dense.

Here, "dense" means "dense in the space of all entire functions in the topology of uniform convergence on compact sets". The translate f_τ of f by a complex number τ is defined by $f_\tau(z) = f(z-\tau)$. By an (n-fold) iterated integral $f^{(-n)}$ of f, we mean a function F such that the n-th derivative $F^{(n)} = f$. For (c), it is clear that some non-trivial choice of the constants of integration must be made—if they were all 0, then the integrals would all vanish at the origin. For iterates,

$$f^{[1]} = f \quad \text{and} \quad f^{[n+1]}(z) = f(f^{[n]}(z)) .$$

We will give a brief resumé of the history of this topic. In [GRM], Gerald MacLane produced an entire function universal in sense (a). His construction was unwittingly rederived by the authors in [BLR]. In [BIR], Birkhoff found an entire function universal in sense (b). Seidel and Walsh later used another construction in [SEW]. So far as we know, no one has previously considered condition (c). There are two related papers by Wolfgang Luh (see [LUH, I] and [LUH, II]). We shall give a simple proof that there does not exist any entire function f that is universal in sense (d).

The main purpose of this paper is to produce a single entire function f that is universal in all three of the senses (a), (b), and (c) at the same time. The mathematics involved is not difficult—just the analyst's usual tools (especially Runge's theorem (see [SAZ]))—but the existence of such an f is an entertaining fact, and working through the proofs has real pedagogical value. It is the second author's experience that students find these results attractive and somewhat paradoxical. We now prove that there exists a triply universal entire function f.

LEMMA. *Given two disjoint closed discs* A *and* B *and an entire function* f, *and* $\varepsilon > 0$ *and a positive integer* n, *there exists a polynomial* P *such that*

$$| P(z) - f(z) | \leqslant \varepsilon , \qquad z \in B ,$$
$$| P^{(j)}(z) | \leqslant \varepsilon , \qquad z \in A, \quad j = 0, 1, ..., n .$$

Proof. (A simple application of Runge's theorem.) Let A' be a closed disc concentric with A, with radius$(A') = \delta +$ radius(A), say, $0 < \delta < 1$, such that $A' \cap B = \emptyset$. Take $\eta > 0$ to be chosen later, and use Runge's theorem to find a polynomial P such that

$$| P(z) - f(z) | \leqslant \varepsilon \qquad \text{for} \quad z \in B ,$$
$$| P(z) | \leqslant \eta \qquad \text{for} \quad z \in A' .$$

Then for $z \in A$ and $j = 0, 1, ..., n$,

$$| P^{(j)}(z) | = \left| \frac{j!}{2\pi i} \int_{\zeta \in \partial A'} \frac{P(\zeta)}{(\zeta - z)^{j+1}} \, d\zeta \right|$$
$$\leqslant \frac{n!}{\delta^{n+1}} (\text{radius}(A'))\eta ,$$

and we may choose η so small that this last expression is less than ε.

Now we construct our function f. We begin with a function $U(z)$ which is universal in sense (a). Let h_j, $j = 1, 2, ...$ be a dense sequence of entire functions (e.g. the polynomials with rational coefficients, in some enumeration). We construct inductively polynomials P_j, closed discs A_j, B_j, and natural numbers n_j. The desired f will be $f(z) = U(z) + \sum_{j=1}^{\infty} P_j(z)$.

The choice of P_1, A_1, B_1, n_1 is not important. We may take $P_1(z) = z$, $A_1 = \{z : | z | \leqslant 1\}$, $B_1 = \{z : | z - 4 | \leqslant 1\}$, and $n_1 = 1$.

Next we specify the induction step. The disc A_{M+1} contains A_j, B_j for $1 \leqslant j \leqslant M$, and also contains $\{z : |z| \leqslant M+1\}$. The disc B_{M+1} is disjoint from A_{M+1} and has radius $M+1$. Let d_M be the maximum of the degrees of $P_1, \ldots, P_M$. Choose $n_{M+1} > n_M$ so that for each i with $1 \leqslant i \leqslant M+1$, there is a j with $d_M < j \leqslant n_{M+1}$ such that $|U^{(j)}(z) - h_i(z)| \leqslant 2^{-(M+1)}$ for $|z| \leqslant M+1$. We use the lemma to obtain P_{M+1} satisfying

$$|h_{M+1}(z-\alpha) - [U(z) + \sum_{j=1}^{M+1} P_j(z)]| < 2^{-(M+1)} \quad \text{for} \quad z \in B_{M+1},$$

where α is the center of B_{M+1}, and

$$|P_{M+1}^{(j)}(z)| \leqslant 2^{-(M+1)} \quad \text{for} \quad z \in A_{M+1}, 0 \leqslant j \leqslant n_{M+1}.$$

This completes the construction of the P_j and the definition of f. Since $\sum_{j=M}^{\infty} |P_j(z)| \leqslant 2^{-M+1}$ for $|z| \leqslant M$, f is indeed an entire function.

If $1 \leqslant i \leqslant M$, then there exists a j with $d_M < j \leqslant n_{M+1}$ with $|U^{(j)}(z) - h_i(z)| \leqslant 2^{-(M+1)}$ for $|z| \leqslant M+1$, so that for $|z| \leqslant M$,

$$\begin{aligned} |f^{(j)}(z) - h_i(z)| &= |U^{(j)}(z) + \sum_{l=M+1}^{\infty} P_l^{(j)}(z) - h_i(z)| \\ &\leqslant 2^{-(M+1)} + \sum_{l=M+1}^{\infty} 2^{-l}. \end{aligned}$$

This establishes that the derivatives of f are dense.

Finally, if $|z| \leqslant M$ and α is the center of B_M, then

$$\begin{aligned} |f(z+\alpha) - h_M(z)| &\leqslant |U(z+\alpha) + \sum_{j=1}^{M} P_j(z+\alpha) - h_M(z)| \\ &\quad + \sum_{j=M+1}^{\infty} |P_j(z+\alpha)| \leqslant 2^{-M} + \sum_{j=M+1}^{\infty} 2^{-j}, \end{aligned}$$

so that the translates of f are dense also.

The next proposition completes the proof that f is triply universal. First, a definition and a lemma:

Notation. Let f be an entire function. Define $I^0 f = f$ and $(I^{n+1} f)(z) = \int_0^z (I^n f)(w)\, dw$.

LEMMA. *For any entire f and any $k > 0$,*

$$\lim_{n\to\infty} \left(\sup_{|z| \leqslant k} |(I^n f)(z)| \right) = 0 .$$

The proof is quite easy—see [BLR].

PROPOSITION. *There exist constants $C_1, C_2, C_3, \ldots$ such that, for any entire function f, if we define the functions Q_k by*

$$\begin{aligned} Q_0(z) &= f(z), \\ Q_1(z) &= (I^1 f)(z) + C_1, \\ Q_2(z) &= (I^2 f)(z) + C_1 z + C_2, \\ &\vdots \\ Q_k(z) &= (I^k f)(z) + \sum_{j=0}^{k} \frac{1}{j!} C_{k-j} z^j, \end{aligned}$$

then the sequence (Q_k) is dense.

Proof of Proposition. Let $h_i(z)$, $i = 1, 2, \ldots$ be a dense sequence of polynomials. We construct the C_i in steps. At step n we have constructed $C_1, C_2, \ldots, C_{L_n}$. For notational convenience we will write $C_0 = 0$.

Step 1: $L_1 = 1$ and $C_1 = 0$.

Step $n+1$: Let $g(z) = g_n(z) = \sum_{j=0}^{L_n} \frac{1}{j!} C_{L_n - j} z^j$.

Choose $M > 0$ so that if $k \geqslant M$ then $|(I^k g)(z)| \leqslant \dfrac{1}{n+1}$ for all $|z| \leqslant n+1$. This is possible by the lemma. Let $L_{n+1} = L_n + M + 1 + d_{n+1}$, where d_{n+1} is the degree of h_{n+1}. Choose $C_{L_n+1}, \ldots, C_{L_{n+1}}$ so that

$$(*) \qquad Q_{L_{n+1}}(z) = (I^{L_{n+1}} f)(z) + (I^{L_{n+1} - L_n} g)(z) + h_{n+1}(z) .$$

To see that this is possible, note that

$$(I^{L_{n+1} - L_n} g)(z) = \sum_{j = L_{n+1} - L_n}^{L_{n+1}} \frac{1}{j!} C_{L_{n+1} - j} z^j ,$$

so that (*) asserts

$$\sum_{j=0}^{L_{n+1}} \frac{1}{j!} z^j C_{L_{n+1} - j} = \sum_{j = L_{n+1} - L_n}^{L_{n+1}} \frac{1}{j!} z^j C_{L_{n+1} - j} + h_{n+1}(z) .$$

In the range $L_{n+1} - L_n \leqslant j \leqslant L_{n+1}$, the coefficients of the z^j are automatically correct. For the range $0 \leqslant j < L_{n+1} - L_n$ we can choose the $C_{L_{n+1} - j}$

to make (*) correct. We note that this is equivalent to choosing C_l for $L_n + 1 \leqslant l \leqslant L_{n+1}$. This completes the construction.

To verify that (Q_k) is dense, let E be an entire function, $N > 0$ and $\varepsilon > 0$. Choose n so that

i) $n > N$,

ii) $| h_n(z) - E(z) | \leqslant \frac{1}{3} \varepsilon$ for $|z| \leqslant N$,

iii) $\frac{1}{n} \leqslant \frac{1}{3} \varepsilon$, and

iv) $| (I^{L_n} f)(z) | \leqslant \frac{1}{3} \varepsilon$ for $|z| \leqslant N$.

Then $| Q_{L_n}(z) - E(z) | \leqslant \varepsilon$ for all $|z| \leqslant N$.

We conclude with a result communicated to us by I. N. Baker, namely that there is no entire function f such that the sequence of its compositional iterates $f^{[n]}$ spans a dense set. If f is linear, then the closed span of its iterates contains only linear functions. (That is, the only univalent entire functions are the linear ones. See [BUR], §11.19, p. 370 and the Notes on p. 407 for several proofs.) If f is not linear, then there exist z_1, z_2 with $z_1 \neq z_2$ but $f(z_1) = f(z_2)$. This equality also holds for all linear combinations of iterates, so the closed span of the iterates lies in the proper closed subspace $\{g : g(z_1) = g(z_2)\}$. Finally, we remark that if f is universal in sense (b) then the pair $f(z)$, $z - 1$ *is* universal under composition.

REFERENCES

[BIR] BIRKHOFF, G. D. Démonstration d'un théorème élémentaire sur les fonctions entières. *C. R. Acad. Sci. Paris 189* (1929), 473-475.

[BLR] BLAIR, Charles and Lee RUBEL. A universal entire function. *Amer. Math. Monthly 90* (1983), 331-332—see also Addendum, to appear December 1983.

[BUR] BURCKEL, Robert G. *An Introduction to Classical Complex Analysis.* Vol. I, Academic Press, New York and San Francisco, 1979.

[GRM] MACLANE, Gerald. Sequences of derivatives and normal families. *J. d'Analyse Math. 2* (1952), 72-87.

[LUH, I] LUH, Wolfgang. On universal functions. *Fourier Analysis and Approximation Theory* (Proc. Colloq. Budapest, 1976), Vol. II, 503-511, Colloq. Math. Soc. Janos Bolyai, 19, North-Holland, Amsterdam 1978.

[LUH, II] —— Universalfunktionen in einfach zusammenhängenden Gebieten. *Aequationes Math. 19* (1979), 183-193.

[SAZ] SAKS, S. and A. ZYMUND. *Analytic functions.* 2nd ed., Polish Scientific Publishers 1965.

[SEW] SEIDEL, W. P. and J. L. WALSH. On approximation by Euclidean and non-Euclidean translates of an analytic function. *Bull. Amer. Math. Soc. 47* (1941), 916-920.

(Reçu le 4 octobre 1983)

Charles Blair

Department of Business Administration
University of Illinois
Urbana, Illinois 61801

Lee Rubel

Department of Mathematics
University of Illinois
Urbana, Illinois 61801