FOUR CHARACTERIZATIONS OF REAL RATIONAL DOUBLE POINTS

Autor(en): **Durfee, Alan H.**

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FOUR CHARACTERIZATIONS OF REAL RATIONAL DOUBLE POINTS [1])

by Alan H. DURFEE

This note is a sequel to the author's paper "Fifteen characterizations of rational double points and simple critical points" [Durfee]. The characterizations of that paper are for complex varieties and complex functions, and involve the Dynkin diagrams A_k, D_k and E_k. It turns out that the missing Dynkin diagrams B_k, C_k, and F_4 (but not G_2) correspond to real singularities and real functions, and that a smaller number of similar characterizations are true for these as well. Four such characterizations are given in Theorem 2 below; the corresponding characterizations for complex functions are recalled in Theorem 1. Theorem 2 combines the work of Arnold and Lipman, but the proofs given here are as elementary as possible, and proceed by direct computation in the various cases or by refering to Theorem 1.

Let $\mathbf{K}$ be either the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. Throughout this note we will be discussing germs at the origin $\mathbf{0}$ of $\mathbf{K}$-analytic functions f of three variables satisfying $f(\mathbf{0}) = 0$. Let $\mathcal{F}$ be the set of all such germs. There is a projection of $\mathcal{F}$ to the set $\mathcal{F}_k$ of k-jets. The set $\mathcal{F}$ has a natural topology defined by letting a basis of open sets be the inverse images of open sets in $\mathcal{F}_k$, for all k. Two such germs f and g are *right equivalent* if there is a germ h of an analytic automorphism of $(\mathbf{K}^3, \mathbf{0})$ such that $fh = g$. The germs f and g are *right-left* equivalent if there is a germ h of an analytic automorphism of $(\mathbf{K}^3, \mathbf{0})$ and a germ k of an analytic automorphism of $(\mathbf{K}, \mathbf{0})$ such that $kfh = g$. A germ f is (right) *simple* if there is a neighborhood of f in $\mathcal{F}$ intersecting only finitely many right equivalence classes of germs.

Let $\mathbf{K} = \mathbf{C}$, and suppose that f has an isolated critical point at $\mathbf{0}$. Then $f^{-1}(0)$ is an analytic variety with an isolated singularity at $\mathbf{0}$. Let $\pi : M \to f^{-1}(0)$ be a resolution of this singularity. The singularity is *rational* if $H^1(M_{\mathcal{O}_M}) = 0$. For further discussion of these topics, see [Durfee].

Four of the fifteen characterizations from [Durfee] of complex germs f have obvious analogues in the real case. These complex characterizations are stated below.

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[1]) This article has already been published in *Nœuds, tresses et singularités*, Monographie de l'Enseignement Mathématique N° 31, Genève 1983, p. 123-128.

THEOREM 1. *Let* $f : (\mathbf{C}^3, \mathbf{0}) \to (\mathbf{C}, 0)$ *be the germ at the origin of a complex analytic function. The following statements are equivalent.*

1. *The germ* f *is right (or right-left) equivalent to one of the germs listed in column* (a) *of the table* (Characterization B1).
2. *The germ* f *is simple* (Characterization B3).
3. *The complex variety* $f^{-1}(0)$ *has a rational singularity at* $\mathbf{0}$ (Characterization A2).
4. *A resolution of the complex variety* $f^{-1}(0)$ *is listed in column* (b) *of the table* (Characterization A3).

The notation for column (b) of the table is as follows: The graph is the dual graph of the exceptional set in a resolution of the variety $f^{-1}(0)$. Each vertex represents a nonsingular rational curve of self-intersection -2, and an edge joining two vertices means that the corresponding curves intersect in one point.

The proof connecting these characterizations is non-trivial; the reader is referred to [Durfee] for details. Note that all the germs in column (a) have real coefficients.

It would be interesting to have some better relations between statements (1) and (3) above. For example, is the genus always less than or equal to the modality? (The proof of a theorem due to Randell (Theorem 11.1 in [Durfee]) unfortunately contained a gap. See also [Wahl, Corollary 4.8].)

Here is the corresponding theorem for real germs:

THEOREM 2. *Let* $f : (\mathbf{R}^3, \mathbf{0}) \to (\mathbf{R}, 0)$ *be the germ at the origin of a real analytic function. The following statements are equivalent:*

1A. *The germ* f *is right-left equivalent to one of the germs listed in column* (a′) *of the table.*

1B. *The germ* f *is right equivalent over* $\mathbf{C}$ *to one of the germs listed in column* (a) *of the table.*

2. *The germ* f *is simple.*

3. *The complexified variety* $f^{-1}(0)$ *has a rational singularity at* $\mathbf{0}$.

4A. *A resolution of the real variety* $f^{-1}(0)$ *is listed in column* (b′) *of the table.*

4B. *A resolution of the complexified variety* $f^{-1}(0)$ *is listed in column* (b) *of the table.*

The notation for column (b′) of the table is as follows: The graph is the dual graph of the exceptional set of a real form of a resolution of the complexified variety $f^{-1}(0)$. Each vertex represents the real form of a complex curve. Two vertices are joined by a solid edge if the corresponding complex curves intersect in one real point, and by a dotted edge if they intersect in two (distinct) conjugate complex points. More specifically, the symbols mean the following:

$\overset{1}{\bullet}$ A nonsingular complex rational curve whose real points are homeomorphic to the circle (e.g., $x^2 + y^2 = 1$ in $\mathbf{C}^2$).

$\overset{1}{\circ}$ A nonsingular complex rational curve with no real points (e.g., $x^2 + y^2 = -1$).

$\overset{2}{\bullet}$ Two conjugate nonsingular rational complex curves intersecting at their one real point (e.g., $x^2 + y^2 = 0$)

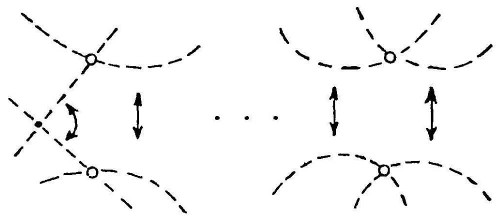
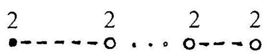
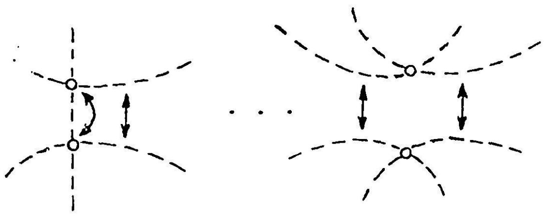
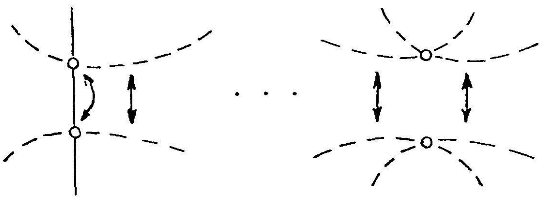
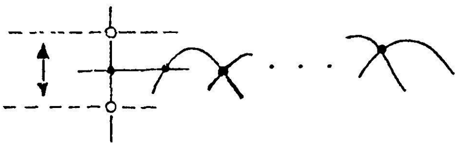
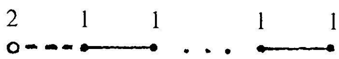
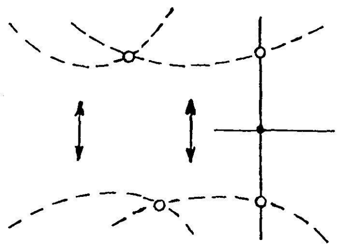
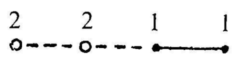
$\overset{2}{\circ}$ Two conjugate disjoint nonsingular rational complex curves with no real points.

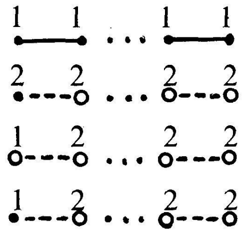
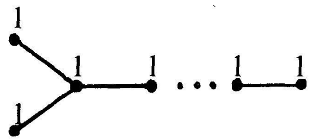
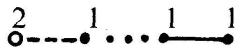
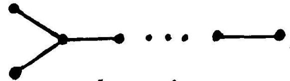
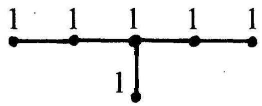
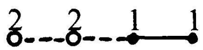
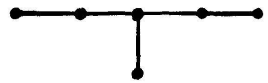
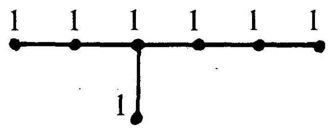
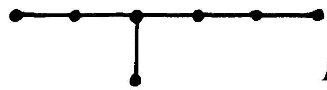
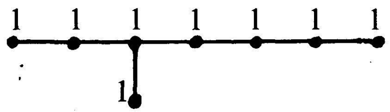
Proof of Theorem 2. Characterizations 1B, 3 and 4B are equivalent by Theorem 1.

(1A⇔1B). Clearly the germs in column (a′) are right equivalent over $\mathbf{C}$ to the corresponding germ in column (a). Conversely, given a real germ f which is right equivalent over $\mathbf{C}$ to a germ in column (a), its corank and codimension as complex germ are known; these have the same value for the real germ. One must then work through the classification theorem [Arnold 1, Siersma] and show that f is right-left equivalent over $\mathbf{R}$ to one of the corresponding germs in column (a′).

(1A⇔2). By [Arnold 1], a simple germ is right-equivalent to one of the germs in column (a), except with a sprinkling of $\pm$ signs. An easy computation shows that these germs are all right-left equivalent to the germs listed in column (a′). Conversely, Arnold proves that all these germs are simple.

(4A⇔4B). Clearly the real resolution in column (b′) has the corresponding complex form in column (b). Conversely, column (b′) is obtained from column (b) by classifying the action of conjugation on the exceptional sets whose dual graph is in column (b). One useful fact is that if a non-singular complex rational curve is taken to itself by conjugation, it must be of type $\overset{1}{\bullet}$ or $\overset{1}{\circ}$: Since the complex curve is nonsingular, so is its real form; this must then be a real one-dimensional manifold with at most one topological component, by Harnack's theorem. The classification of involutions is not difficult, and details will be omitted. The nontrivial involutions on the exceptional sets are as follows (where a solid line is a complex rational curve whose real points are homeomorphic to the circle, a dotted line is a curve with at most one real point, $\bullet$ denotes a real point of intersection, and $\circ$ denotes a non-real complex point of intersection):

exceptional set with conjugation	dual graph of resolution in column (b′)
	
	
	
	
	

(a) complex germ $f(x, y, z)$	(a′) real germ $f(x, y, z)$	(b′) resolution of real variety $f^{-1}(0)$		(b) resolution of complex variety $f^{-1}(0)$	
$x^{k+1} + y^2 + z^2$ $(k \geqslant 1)$	$x^{k+1} + y^2 - z^2$ $x^{k+1} + y^2 + z^2 \begin{cases} k \text{ even} \\ k \text{ odd} \end{cases}$ $-x^{k+1} + y^2 + z^2$, k odd		k vertices $k/2$ vertices $(k+1)/2$ vertices $(k+1)/2$ vertices	 k vertices	A_k
$x^{k-1} + xy^2 + z^2$ $(k \geqslant 4)$	$-x^{k-1} + xy^2 + z^2$ $x^{k-1} + xy^2 + z^2$	 	k vertices $k-1$ vertices	 k vertices	D_k
$x^3 + y^4 + z^2$	$x^3 - y^4 + z^2$ $x^3 + y^4 + z^2$	 			E_6
$x^3 + xy^3 + z^2$	$x^3 + xy^3 + z^2$				E_7
$x^3 + y^5 + z^2$	$x^3 + y^5 + z^2$				E_8

Finally, it remains to verify that the equations $f(x, y, z) = 0$ in column (a′) have the resolutions of column (b′). This can be verified by blowing up points in three-space. (Note that by (4B⇔4A) above, the graphs are already known, and it is only necessary to match them with the equations.) The details are hard to write down, and will be omitted. This completes the proof of Theorem 2.

It would be interesting to understand the resolution of real singularities better. For example, what does the resolution of $x^3 + y^2 - z^2 = 0$ look like, topologically? It would also be interesting to understand the connections (if any) between the modality of a real germ as real germ and as complex germ.

More generally, the Dynkin diagrams B_k, C_k, and F_4 arise in situations where there is an involution on an object corresponding to the diagrams A_k, D_k and E_k. In the above theorem, the involution is conjugation. Connections with simple algebraic groups are discussed in [Slodowy, 6.2 and Appendix I]. Another example is critical points of functions on manifolds with boundary [Arnold 2]; these correspond to functions on the doubled manifold invariant with respect to the obvious involution. Lastly, the diagram G_2 arises where there is an automorphism of order 3 on an object corresponding to D_4.

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Alan H. Durfee

Smith College
Northampton, MA 01063