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REPRESENTATION OF REAL NUMBERS BY MEANS OF FIBONACCI NUMBERS

by Paulo Ribenboim

Dedicated to Professor D.H. Lehmer at the occasion of his eightieth birthday.

The purpose of this note is to derive a new representation of positive real numbers as sums of series involving Fibonacci numbers. This will be an easy application of an old result of Kakeya [4]. The paper concludes with a result of Landau [5], relating the sum $\sum_{n=1}^{\infty} \frac{1}{F_n}$ with values of theta series; we believe it worthwhile to unearth Landau's result, which is now rather inaccessible.

1. Let $(s_i)_{i \geq 1}$ be a sequence of positive real numbers, such that $s_1 > s_2 > s_3 > \ldots$ and $\lim_{i \to \infty} s_i = 0$. Let $S = \sum_{i=1}^{\infty} s_i \leq \infty$.

We say that $x > 0$ is representable by the sequence $(s_i)_{i \geq 1}$ if $x = \sum_{j=1}^{\infty} s_{i_j}$ (with $i_1 < i_2 < i_3 < \ldots$). Then necessarily $x \leq S$.

The first result is due to Kakeya; for the sake of completeness, we give a proof:

Proposition 1. *The following conditions are equivalent:*

1) *Every* $x, 0 < x \leq S$, *is representable by the sequence* $(s_i)_{i \geq 1}$, $x = \sum_{j=1}^{\infty} s_{i_j}$, *where* i_1 *is the smallest index such that* $s_{i_1} < x$.

2) *Every* $x, 0 < x < S$ *is representable by the sequence* $(s_i)_{i \geq 1}$.

3) *For every* $n \geq 1, s_n \leq \sum_{i=n+1}^{\infty} s_i$.

Proof. $1 \to 2$. This is trivial.

$2 \to 3$. If there exists $n \geq 1$ such that $s_n > \sum_{i=n+1}^{\infty} s_i$, let x be such that $s_n > x > \sum_{i=n+1}^{\infty} s_i$. By hypothesis, $x = \sum_{j=1}^{\infty} s_{i_j}$ with $i_1 < i_2 < \ldots$ Since $s_n > x > s_{i_1}$, then $n < i_1$, hence $x = \sum_{j=1}^{\infty} s_{i_j} \leq \sum_{k=n+1}^{\infty} s_k$, which is absurd.

$3 \to 1$. Since $\lim_{i\to\infty} s_i = 0$, there exists the smallest index i_1 such that $s_{i_1} < x$.

Similarly, there exists the smallest index i_2 such that $i_1 < i_2$ and $s_{i_2} < x - s_{i_1}$.

More generally, for every $n \geqslant 1$ we define i_n to be the smallest index such that $i_{n-1} < i_n$ and $s_{i_n} < x - \sum_{j=1}^{n-1} s_{i_j}$.

Then $x \geqslant \sum_{j=1}^{\infty} s_{i_j}$. Suppose that $x > \sum_{j=1}^{\infty} s_{i_j}$.

We note that there exists N such that if $m \geqslant N$ then $s_{i_m} < x - \sum_{j=1}^{m} s_{i_j}$. Otherwise, there exist infinitely many indices $n_1 < n_2 < n_3 < \ldots$ such that $s_{i_{n_k}} \geqslant x - \sum_{j=1}^{n_k} s_{i_j}$. At the limit, we have $0 = \lim_{k\to\infty} s_{i_{n_k}} \geqslant x - \sum_{j=1}^{\infty} s_{i_j} > 0$, and this is a contradiction.

We choose N minimal with above property.

We show: for every $m \geqslant N$, $i_m + 1 = i_{m+1}$. In fact

$$s_{i_m+1} < s_{i_m} < x - \sum_{j=1}^{m} s_{i_j},$$

so by definition of the sequence of indices, $i_m + 1 = i_{m+1}$. Therefore the following sets coincide: $\{i_N, i_N+1, i_N+2, \ldots\} = \{i_N, i_{N+1}, i_{N+2}, \ldots\}$.

Next we show that $i_N = 1$. If $i_N > 1$ we consider the index $i_N - 1$, and by hypothesis (3):

$$s_{i_N-1} \leqslant \sum_{k=i_N}^{\infty} s_k = \sum_{j=N}^{\infty} s_{i_j} < x - \sum_{j=1}^{N-1} s_{i_j}.$$

We have $i_{N-1} \leqslant i_N - 1 < i_N$. If $i_{N-1} < i_N - 1$ this is impossible, because i_N was defined to be the smallest index such that $i_{N-1} < i_N$ and $s_{i_N} < x - \sum_{j=1}^{N-1} s_{i_j}$. Thus $i_{N-1} = i_N - 1$, that is $s_{i_{N-1}} < x - \sum_{j=1}^{N-1} s_{i_j}$ and this is against the choice of N as minimal with the property indicated.

Thus $i_N = 1$ and $x > \sum_{j=1}^{\infty} s_{i_j} = \sum_{i=1}^{\infty} s_i = S$, against the hypothesis. □

We remark now that if the above conditions are satisfied for the sequence $(s_i)_{i\geqslant 1}$, if $m \geqslant 0$ then every x, $0 < x < S' = \sum_{i=m+1}^{\infty} s_i$ is representable by the sequence $(s_i)_{i\geqslant m+1}$ with i_1 the smallest index such that $m + 1 \leqslant i_1$ and $s_{i_1} < x$.

Indeed, condition (3) holds for $(s_i)_{i \geqslant 1}$ hence also for $(s_i)_{i \geqslant m+1}$. Since $0 < x < S'$, the remark follows from the proposition.

Proposition 1 has been generalized (see for example Fridy [3]). Now we consider the question of unique representation (this was generalized by Brown in [1]).

PROPOSITION 2. *With above notations, the following conditions are equivalent:*

2′) *Every* $x, 0 < x < S,$ *has a unique representation* $x = \sum_{j=1}^{\infty} s_{i_j}$.

3′) *For every* $n \geqslant 1, s_n = \sum_{i=n+1}^{\infty} s_i$.

4′) *For every* $n \geqslant 1, s_n = \frac{1}{2^{n-1}} s_1$ *(hence* $S = 2s_1$*).*

Proof. $2' \to 3'$. Suppose there exists $n \geqslant 1$ such that $s_n \neq \sum_{i=n+1}^{\infty} s_i$. Since (2′) implies (2) hence also (3) then $s_n < \sum_{i=n+1}^{\infty} s_i$. Let x be such that $s_n < x < \sum_{i=n+1}^{\infty} s_i$. By the above remark, x is representable by the sequence $\{s_i\}_{i \geqslant n+1}$, that is $x = \sum_{\substack{j=1 \\ k_j \geqslant n+1}}^{\infty} s_{k_j}$. On the other hand, (2′) implies (2), hence also (1) and x has a representation $x = \sum_{j=1}^{\infty} s_{i_j}$, where i_1 is the smallest index such that $s_{i_1} < x$. From $s_n < x$ it follows that $i_1 \leqslant n$ and so x would have two distinct representations, against the hypothesis.

$3' \to 4'$. We have $s_n = s_{n+1} + \sum_{i=n+2}^{\infty} s_i = 2s_{n+1}$ for every $n \geqslant 1$ hence $s_n = \frac{1}{2^{n-1}} s_1$ for every $n \geqslant 1$.

$4' \to 2'$. Suppose that there exists $x, 0 < x < S$ having two distinct representations

$$x = \sum_{j=1}^{\infty} s_{i_j} = \sum_{j=1}^{\infty} s_{k_j}.$$

Let j_0 be the smallest index such that $i_{j_0} \neq k_{j_0}$, say $i_{j_0} < k_{j_0}$. Then $\sum_{j=j_0}^{\infty} s_{i_j} = \sum_{j=j_0}^{\infty} s_{k_j} \leqslant \sum_{n=i_{j_0}+1}^{\infty} s_n$.

By hypothesis, after dividing by s_1, we have

$$\sum_{n=i_{j_0}}^{\infty} \frac{1}{2^n} \geqslant \sum_{j=j_0}^{\infty} 2^{1-k_j} = \sum_{j=j_0}^{\infty} 2^{1-i_j}$$

$$= 2^{1-i_{j_0}} + \sum_{j=j_0+1}^{\infty} 2^{1-i_j} = \sum_{n=i_{j_0}}^{\infty} 2^{-n} + \sum_{j=j_0+1}^{\infty} 2^{1-i_j}$$

hence $\sum_{j=j_0+1}^{\infty} 2^{1-i_j} \leqslant 0$, which is impossible. □

For practical applications, we note:
If $s_n \leqslant 2s_{n+1}$ for every $n \geqslant 1$ then condition (3) is satisfied.
Indeed

$$\sum_{i=n+1}^{\infty} s_i \leqslant 2 \sum_{i=n+1}^{\infty} s_{i+1} = 2 \sum_{i=n+2}^{\infty} s_i, \quad \text{hence} \quad s_{n+1} \leqslant \sum_{i=n+2}^{\infty} s_i$$

and $s_n \leqslant 2\, s_{n+1} \leqslant \sum_{i=n+1}^{\infty} s_i$.

2. Now we give various ways of representing real numbers.

First, the dyadic representation, which may of course be easily obtained directly:

COROLLARY 1. *Every real number* x, $0 < x < 1$, *may be written uniquely in the form* $x = \sum_{j=1}^{\infty} \frac{1}{2^{n_j}}$ *(with* $1 \leqslant n_1 < n_2 < n_3 < \ldots$*).*

Proof. This has been shown in proposition 2, taking $s_1 = \frac{1}{2}$. □

COROLLARY 2. *Every positive real number* x *may be written in the form* $x = \sum_{j=1}^{\infty} \frac{1}{n_j}$ *(with* $n_1 < n_2 < n_3 < \ldots$*).*

Proof. We consider the sequence $\left(\frac{1}{n}\right)_{n \geqslant 1}$, which is decreasing with limit equal to zero, and we note that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and $\frac{1}{n} \leqslant \frac{2}{n+1}$ for every $n \geqslant 1$.

Thus by Kakeya's theorem and the above remark, every $x > 0$ is representable as indicated. □

COROLLARY 3. *Every positive real number x may be written as* $x = \sum_{j=1}^{\infty} \frac{1}{p_{i_j}}$ *(where $p_1 < p_2 < p_3 < \ldots$ is the increasing sequence of prime numbers).*

Proof. We consider the sequence $\left(\frac{1}{p_i}\right)_{i \geq 1}$, which is decreasing with limit equal to zero. As Euler proved $\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty$. By Tschebycheff's theorem (proof of Bertrand's "postulate") there is a prime in each interval $(n, 2n)$; thus $p_{i+1} < 2\,p_i$ and $\frac{1}{p_i} < \frac{2}{p_{i+1}}$ for every $i \geq 1$. By Kakeya's theorem and the above remark, every $x > 0$ is representable as indicated. □

3. Now we shall represent real numbers by means of Fibonacci numbers and we begin giving some properties of these numbers.

The Fibonacci numbers are: $F_1 = F_2 = 1$ and for every $n \geq 3$, F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$.

Thus the sequence of Fibonacci numbers is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

In the following proposition, we give a closed form expression for the Fibonacci numbers; this is due to Binet (1843).

Let $\alpha = \frac{\sqrt{5}+1}{2}$ (the golden number) and $\beta = -\frac{\sqrt{5}-1}{2}$, so $\alpha + \beta = 1$, $\alpha\beta = -1$, thus α, β are the roots of $X^2 - X - 1 = 0$ and $-1 < \beta < 0 < 1 < \alpha$.

We have

LEMMA 1. *For every $n \geq 1$,* $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ *and* $\frac{\alpha^{n-1}}{\sqrt{5}} < F_n < \frac{\alpha^{n+1}}{\sqrt{5}}$.

Proof. We consider the sequence of numbers $G_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ for $n \geq 1$.

Then $G_1 = G_2 = 1$; moreover

$$G_{n-1} + G_{n-2} = \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} + \frac{\alpha^{n-2} - \beta^{n-2}}{\sqrt{5}}$$

$$= \frac{\alpha^{n-2}(\alpha+1) - \beta^{n-2}(\beta+1)}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\sqrt{5}} = G_n,$$

because $\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$. Therefore the sequence $(G_n)_{n\geqslant 1}$ coincides with the Fibonacci sequence.

Now we establish the estimates.

If $n \geqslant 1$ then $(-\beta)^n = \dfrac{1}{\alpha^n} < \alpha^{n-1} = -\alpha^n\beta = \alpha^n(\alpha-1) = \alpha^{n+1} - \alpha^n$, so

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \leqslant \frac{\alpha^n + (-\beta)^n}{\sqrt{5}} < \frac{\alpha^{n+1}}{\sqrt{5}}.$$

Similarly, if $n \geqslant 2$ then $(-\beta)^n = \dfrac{1}{\alpha^n} < \alpha^{n-2} = -\alpha^{n-1}\beta = \alpha^{n-1}(\alpha-1)$ $= \alpha^n - \alpha^{n-1}$ so $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}} \geqslant \dfrac{\alpha^n - (-\beta)^n}{\sqrt{5}} > \dfrac{\alpha^{n-1}}{\sqrt{5}}$; this is also true when $n = 1$. □

For every $m \geqslant 1$ let $I_m = \sum_{n=1}^{\infty} \dfrac{1}{F_n^{1/m}}$.

We have:

Lemma 2. *For every* $m \geqslant 1$, $I_m < \infty$, $I_1 < I_2 < I_3, \ldots,$ *and* $\lim_{m\to\infty} I_m = \infty$.

Proof. We have

$$I_m < \sum_{n=1}^{\infty} \left(\frac{\sqrt{5}}{\alpha^{n-1}}\right)^{1/m} = (\sqrt{5})^{1/m} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha^{1/m}}\right)^{n-1} = \frac{(\sqrt{5})^{1/m}\,\alpha^{1/m}}{\alpha^{1/m} - 1},$$

noting that $\dfrac{1}{\alpha^{1/m}} < 1$.

Next, we have

$$I_{m-1} = \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m-1}} < \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m}} = I_m.$$

Finally

$$I_m = \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m}} > \sum_{n=1}^{\infty} \left(\frac{\sqrt{5}}{\alpha^{n+1}}\right)^{1/m} = \frac{(\sqrt{5})^{1/m}}{\alpha^{1/m}} \times \frac{1}{\alpha^{1/m} - 1};$$

thus $\lim_{m\to\infty} I_m = \infty$. □

PROPOSITION 3. *For every positive real number* x *there exists a unique* $m \geqslant 1$ *such that* $x = \sum_{j=1}^{\infty} \frac{1}{F_{i_j}^{1/m}}$, *but* x *is not of the form* $\sum_{j=1}^{\infty} \frac{1}{F_{i_j}^{1/(m-1)}}$.

Proof. First we note that each of the sequences $\left(\frac{1}{F_n^{1/m}}\right)_{n \geqslant 1}$ is decreasing with limit equal to zero. By the above proposition, there exists $m \geqslant 1$ such that $I_{m-1} < x \leqslant I_m$ (with $I_0 = 0$).

We observe that $\frac{1}{F_n} \leqslant \frac{2}{F_{n+1}} \leqslant \frac{2^m}{F_{n+1}}$ for $m \geqslant 1$, because $F_{n+1} = F_n + F_{n-1} < 2\,F_n$. By proposition 1 and a previous remark, x is representable as indicated, while the last assertion follows from $x > I_{m-1} = \sum_{i=1}^{\infty} \frac{1}{F_i^{1/m-1}}$.

□

The number $I_1 = \sum_{n=1}^{\infty} \frac{1}{F_n}$ appears to be quite mysterious. As we have seen $\sqrt{5} < I_1 < \sqrt{5}\,\frac{\alpha}{\alpha - 1}$.

4. In 1899, Landau gave an expression of I_1 in terms of Lambert series and Jacobi theta series. The Lambert series is $L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}$; it is convergent for $0 < x < 1$, as easily verified by the ratio test.

Jacobi theta series, which are of crucial importance, for example in the theory of elliptic functions, are defined as follows, for $0 < |q| < 1$ and $z \in C$:

$$\theta_1(z, q) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n-\frac{1}{2}\right)^2} e^{(2n-1)\pi i z}$$
$$= 2\,q^{1/4} \sin \pi z - 2\,q^{9/4} \sin 3\pi z + 2\,q^{25/4} \sin 5\pi z - \ldots$$

$$\theta_2(z, q) = \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n-1)\pi i z}$$
$$= 2\,q^{1/4} \cos \pi i z + 2\,q^{9/4} \cos 3\pi z + 2\,q^{25/4} \cos 5\pi z + \ldots$$

$$\theta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i z}$$
$$= 1 + 2q \cos 2\pi z + 2q^4 \cos 4\pi z + 2q^9 \cos 6\pi z + \ldots$$

$$\theta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i z}$$
$$= 1 - 2q \cos 2\pi z + 2q^4 \cos 4\pi z - 2q^9 \cos 6\pi z + \ldots$$

In particular, we have

$$\theta_1(0, q) = 0$$

$$\theta_2(0, q) = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \dots$$

$$\theta_3(0, q) = 1 + 2q + 2q^4 + 2q^9 + \dots$$

$$\theta_4(0, q) = 1 - 2q + 2q^4 - 2q^9 + \dots$$

Now we prove Landau's result:

PROPOSITION 4. *We have:*

1) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right].$$

2) $$\sum_{n=0}^{\infty} \frac{1}{F_{2n-1}} = -\sqrt{5}\,(1+2\beta^4+2\beta^{16}+2\beta^{36}+\dots)(\beta+\beta^9+\beta^{25}+\dots)$$

$$= -\frac{\sqrt{5}}{2}[\theta_3(0, \beta) - \theta_2(0, \beta^4)]\,\theta_2(0, \beta^4).$$

Proof:

1) We have $\dfrac{1}{F_n} = \dfrac{\sqrt{5}}{\alpha^n - \beta^n} = \dfrac{\sqrt{5}}{\dfrac{(-1)^n}{\beta^n} - \beta^n} = \dfrac{\sqrt{5}\,\beta^n}{(-1)^n - \beta^{2n}}$

so

$$\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1-\beta^{4n}} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \beta^{(4k+2)n} = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \beta^{(4k+2)n}$$

$$= \sum_{k=0}^{\infty} \frac{\beta^{4k+2}}{1-\beta^{4k+2}} = \frac{\beta^2}{1-\beta^2} + \frac{\beta^6}{1-\beta^6} + \frac{\beta^{10}}{1-\beta^{10}} + \dots$$

Since $|\beta| < 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\left[L(\beta^2) - L(\beta^4)\right] = \sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right].$$

2) Now we have

$$\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{1}{F_{2n-1}} = -\sum_{n=0}^{\infty}\frac{\beta^{2n+1}}{1+\beta^{4n+2}} = -\sum_{n=0}^{\infty}\beta^{2n+1}(1-\beta^{4n+2}+\beta^{8n+4}+\ldots)$$

$$\begin{aligned} &= (-\beta+\beta^3-\beta^5+\beta^7-\beta^9+\ldots) \\ &+ (-\beta^3+\beta^9-\beta^{15}+\beta^{21}-\ldots) \\ &+ (-\beta^5+\beta^{15}-\beta^{25}+\beta^{35}-\ldots) \\ &+ (-\beta^7+\beta^{21}-\beta^{35}+\beta^{49}-\ldots) + \ldots \end{aligned}$$

Now we need to determine the coefficient of β^m (for m odd) remarking that since the series is absolutely convergent, its terms may be rearranged.

If m is odd and d divides m, then β^m appears in the horizontal line beginning with $-\beta^{\frac{m}{d}}$ with the sign

$$\begin{cases} + & \text{when } d \equiv 3 \pmod 4 \\ - & \text{when } d \equiv 1 \pmod 4 . \end{cases}$$

Thus the coefficient ε_m of β^m is $\varepsilon_m = \delta_3(m) - \delta_1(m)$ where

$$\delta_1(m) = \#\{d \mid 1 \leqslant d \leqslant m, \quad d \mid m \quad \text{and} \quad d \equiv 1 \pmod 4\}$$
$$\delta_3(m) = \#\{d \mid 1 \leqslant d \leqslant m, \quad d \mid m \quad \text{and} \quad d \equiv 3 \pmod 4\}$$

A well-known result of Jacobi (see Hardy & Wright's book, page 241) relates the difference $\delta_1(m) - \delta_3(m)$ with the number $r(m) = r_2(m)$ of representations of m as sums of two squares. Precisely, let $r(m)$ denote the number of pairs (s, t) of integers (including the zero and negative integers) such that $m = s^2 + t^2$. Jacobi showed that

$$r(m) = 4\,[\delta_1(m) - \delta_3(m)]\,.$$

It follows that the number $r'(m)$ of pairs (s, t) of integers with $s > t \geqslant 0$ and $m = s^2 + t^2$ is

$$r'(m) = \begin{cases} \dfrac{r(m)}{8} & \text{when } m \text{ is not a square} \\[2ex] \dfrac{r(m)-4}{8} + 1 = \dfrac{r(m)+4}{8} & \text{when } m \text{ is a square} \end{cases}$$

(the first summand above corresponds to the representation of m as a sum of two non-zero squares).

Therefore

$$\varepsilon_m = -\frac{r(m)}{4} = \begin{cases} -2r'(m) & \text{when } m \text{ is not a square} \\ \\ -(2r'(m)-1) & \text{when } m \text{ is a square}. \end{cases}$$

Since m is odd then $s \not\equiv t \pmod 2$ and therefore

$$\begin{aligned} \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \varepsilon_m \beta^m \\ &= -2(1+\beta^4+\beta^{16}+\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots) \\ &\quad + (\beta+\beta^9+\beta^{25}+\ldots) \\ &= -(1+2\beta^4+2\beta^{16}+2\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots). \end{aligned}$$

So

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = -\sqrt{5}(1+2\beta^4+2\beta^{16}+2\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots).$$

We may now express this formula in terms of Jacobi series. Namely

$$\begin{aligned} 1 + 2\beta^4 + 2\beta^{16} + 2\beta^{36} + \ldots &= (1+2\beta+2\beta^4+2\beta^9+2\beta^{16}+\ldots) \\ &\quad - (2\beta+2\beta^9+2\beta^{25}+\ldots) = \theta_3(0,\beta) - \theta_2(0,\beta^4) \end{aligned}$$

so

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = -\frac{\sqrt{5}}{2}\left[\theta_3(0,\beta) - \theta_2(0,\beta^4)\right]\theta_2(0,\beta^4). \qquad \square$$

An unpublished formula of Gert Almqvist (1983) gives another expression of I_1 only in terms of Jacobi theta series:

$$I_1 = \frac{\sqrt{5}}{4}\left\{\left[\theta_2\left(0, -\frac{1}{\beta^2}\right)\right]^2 + \frac{1}{\pi}\int_0^1 \left(\frac{d}{dx}\log\theta_4\left(x, -\frac{1}{\beta^2}\right)\right)\cot \pi x dx\right\}.$$

Carlitz considered also in 1971 the following numbers:

$$S_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+k}}$$

so $S_0 = \sum_{n=1}^{\infty} \frac{1}{F_n} = I_1$.

Clearly, all the above series are convergent. Carlitz showed that $S_3, S_7, S_{11}, \ldots \in \mathbf{Q}(\sqrt{5})$, while $S_{4k} = r_k + r'_k S_0$ for $k \geqslant 1$ and $r_k, r'_k \in \mathbf{Q}$.

One may ask: what kind of number is S_0? Are the numbers S_0, S_1, S_2 algebraically independent?

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