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Autor: Hiller, Howard
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FLAT MANIFOLDS WITH $\mathbf{Z}/p^2$ HOLONOMY

by Howard HILLER [1])

By a flat manifold X we will always mean a compact, connected Riemannian manifold of constant curvature zero. Each such space X arises as a quotient E^n/Γ where E^n is n-dimensional Euclidean space and $\Gamma = \pi_1(X)$ is a discrete group of isometries acting freely on E^n, (so X is also called locally Euclidean or a Euclidean space form). The group Γ fits into a short exact sequence:

(0.1) $$0 \to M \to \Gamma \to H \to 1$$

where H is the finite holonomy group of X acting faithfully on a free abelian group M of rank n. Furthermore, if N denotes the normalizer of H in $\mathrm{Aut}(M) \cong GL_n(\mathbf{Z})$, then the affine diffeomorphism class of X corresponds precisely to an orbit of N on the "special" classes of $H^2(H, M)$ (see definition preceeding 2.6). Following Charlap [4], we call X an *H-manifold.*

Charlap [4] has given a complete classification of $\mathbf{Z}/p$-manifolds, p a prime number. His results rely on Reiner's description [10] of the integral representation theory of prime order groups.

The success of the classification in this special case depends, in part, on the following result.

THEOREM (A. Jones [9]). *A finite group H admits finitely many isomorphism classes of indecomposable integral representations (i.e. is of* finite representation type*) if and only if all Sylow p-subgroups of H are cyclic of order p or p^2.*

This result suggests the naturality of generalizing Charlap's classification to $\mathbf{Z}/p^2$-manifolds. In particular, such a classification would contribute to the study of H-manifolds, H cyclic of cube-free order, by using appropriate induction techniques.

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The case $H = \mathbf{Z}/p^2$ differs fundamentally from that of $H = \mathbf{Z}/p$ in that the number of genera of indecomposable $\mathbf{Z}/p^2$-lattices is a function of p (in fact, $4p+1$) while there are 3 genera of indecomposable $\mathbf{Z}/p$-lattices, for any prime p. Furthermore it is no longer the case that only the trivial representation admits non-trivial (and "special") two-dimensional cohomology classes. Eventually we restrict to the case $p = 2$ where there are 9 genera (originally described by Roiter [14] correcting a mistake in Diederichsen [6]) and 3 of these admit special classes. The assumption $p = 2$ also insures that the genera are identical to the isomorphism classes so there are no further invariants to consider. This follows from work of Reiner [13] and the fact that $\mathbf{Z}[e^{2\pi i/m}]$ is a unique factorization domain for $m = 2, 4$.

As the smallest dimension of a $\mathbf{Z}/p^2$-manifold is $p^2 - p + 1$ (this is a special case of results from [8]) only the case $p = 2$ produces flat manifolds of dimension 5, the smallest dimension for which one lacks a complete classification. We show that there are at least 16 5-dimensional flat manifolds with holonomy $\mathbf{Z}/4$ and give a general lower bound for any dimension.

The main ingredient for these results is the work of Heller and Reiner [7] on the integral representation theory of $\mathbf{Z}/p^2$, reviewed in section 1. In section 2 we study the cohomology of the indecomposables, compute their restrictions to the subgroup of order p and identify the "special" classes. Finally in section 3 we restrict to the case $p = 2$ and study the class of $\mathbf{Z}/4$-manifolds.

It is hoped that this example of holonomy classification will succeed in exposing the role played by integral representation theory and cohomology of groups in understanding the structure of flat Riemannian manifolds.

It is a pleasure to thank I. Reiner for helpful correspondence concerning integral representation theory.

§ 1. Genera of $\mathbf{Z}/p^2$-lattices

We begin by briefly reviewing the language and philosophy of the integral representation theory of finite groups (see [5], [11]). We then give Heller and Reiner's description [7] of the genera of $\mathbf{Z}/p^2$ lattices as extensions.

Suppose Λ is a $\mathbf{Z}$-order in a $\mathbf{Q}$-algebra A. A Λ-*lattice* is a left Λ-module that is also a free abelian group of finite rank. The basic problem of integral representation theory is the classification of such Λ-lattices, Λ fixed. The **Z**-orders that we will need are group rings of finite groups $\mathbf{Z}G \subset \mathbf{Q}G$ and rings of algebraic integers $\mathcal{O}_K$ in an algebraic number field K. We sometimes refer to a $\mathbf{Z}G$-lattice as a G-lattice.

Let $\mathbf{Z}_p$ (resp. $\mathbf{Q}_p$) denote the p-adic completion of **Z** (resp. **Q**). It is easy to see that $\Lambda_p = \mathbf{Z}_p \otimes \Lambda$ is a $\mathbf{Z}_p$-order in the $\mathbf{Q}_p$-algebra A_p. Furthermore, any Λ-lattice M yields a Λ_p-lattice $M_p = \mathbf{Z}_p \otimes M$. One says that M and M' are *locally isomorphic* (or in the same *genus*) if $M_p \cong M'_p$ as Λ_p-modules for all primes p.

The classification of Λ-lattices is often attacked by a "local-to-global" approach. By this we mean the solution of the following two problems:

1. (local) Determine a complete set of invariants of the genus of a Λ-lattice.

2. (global) Determine a complete set of invariants of the isomorphism class of Λ-lattice within a fixed genus.

This approach has been very successful and the examples below illustrate it.

We introduce some notation. Let ω (resp. ζ) denote a primitive p^2 (resp. p^{th}) root of unity and let $R_1 = \mathbf{Z}[\zeta]$, $R_2 = \mathbf{Z}[\omega]$. We also let $\Lambda_i = \mathbf{Z}[\mathbf{Z}/p^i]$. The classification of lattices over a ring of algebraic integers, or more generally a Dedekind domain is classic, and is a good example of the local-to-global approach.

(1.1) THEOREM (Steinitz). *If R is a Dedekind domain, every R-lattice is a direct sum of non-zero ideals of R. The genus of an R-lattice is determined by the number of non-zero ideals occurring, its* rank. *The isomorphism class of the R-lattice within the genus is determined by the ideal class of the product of the ideals as an element of the ideal class group of R (the* Steinitz class *of the lattice).*

The classification problem for $\mathbf{Z}/p$-lattices was solved by Diederichsen [6] and Reiner [10]. Again the local-to-global approach is useful.

If $\mathfrak{a}$ denotes a non-zero ideal of R_1 let $E(\mathfrak{a})$ denote the non-split extension of $\mathfrak{a}$ by the trivial lattice **Z**. The genus of $\mathfrak{a}$ (resp. $E(\mathfrak{a})$) is denoted α (resp. β). Every $\mathbf{Z}/p$-lattice M can be written:

$$M = \mathbf{Z}^{(a)} \oplus \sum_{i=1}^{b} \mathfrak{a}_i \oplus \sum_{i=1}^{c} E(\mathfrak{a}'_i)\,.$$

The genus of the $\mathbf{Z}/p$-lattice M is determined by the multiplicities a, b, c of the three indecomposable genera $\mathbf{1}$, α, β. The isomorphism class of the lattice within its genus is completely determined by the ideal class $\prod_{i=1}^{b} \mathfrak{a}_i \cdot \prod_{i=1}^{c} \mathfrak{a}'_i$ in the ideal class group of R_1.

The solution of the (local) classification problem for $R_2 = \mathbf{Z}[\omega]$ (a Dedekind domain) and $\Lambda_1 = \mathbf{Z}[\mathbf{Z}/p]$ can be combined to classify genera of $\mathbf{Z}/p^2$-lattices. The technique used is essentially homological. If M is a $\mathbf{Z}/p^2$-lattice, we let $L = \{x \in M : (x^p - 1)M = 0\}$. L is a $\mathbf{Z}/p$-lattice and fits into a $\mathbf{Z}/p^2$-exact sequence

$$0 \to L \to M \to N \to 0$$

where N is an R_2-lattice. Hence one is reduced to classifying extensions of R_2-lattices by $\mathbf{Z}/p$-lattices using homological methods. It is not difficult to show (see [13, p. 478]) that $\operatorname{Ext}_2(R_2, L) \equiv L/pL$, where L is an arbitrary $\mathbf{Z}/p$-lattice. In fact, if $\alpha \in L/pL$ then the corresponding extension is given by the pushout diagram:

$$\text{(1.2)} \qquad \begin{array}{ccccccccc} 0 & \to & \varphi_2\Lambda_2 & \to & \Lambda_2 & \to & R_2 & \to & 0 \\ & & {}^{\alpha}\downarrow & & \downarrow & & \| & & \\ 0 & \to & L & \to & M & \to & R_2 & \to & 0 \end{array}$$

where φ_2 is the cyclotomic polynomial $1 + x^p + x^{2p} + \dots + x^{(p-1)p}$ and the map α is (by abuse of language) the map that sends φ_2 to α. We write (L, α) for this extension. The final result is:

(1.3) THEOREM (Heller and Reiner [7]). *There are* $4p + 1$ *genera of indecomposable* $\mathbf{Z}/p^2$*-lattices given by:*

$$\begin{array}{lll} M_1 = \mathbf{Z}, & & \\ M_2 = R_1, & & \\ M_3 = R_2, & & \\ M_4 = \Lambda_1, & & \\ M_5 = (\mathbf{Z}, 1), & & \\ M_6(k) = (\Lambda_1, \lambda^k), & & 0 \leqslant k \leqslant p-1, \\ M_7(k) = (Z \oplus \Lambda_1, \mathbf{1} \oplus \lambda^k), & & 1 \leqslant k \leqslant p-2, \\ M_8(k) = (R_1, \lambda^k), & & 0 \leqslant k \leqslant p-2, \\ M_9(k) = (Z \oplus R_1, \mathbf{1} \oplus \lambda^k), & & 0 \leqslant k \leqslant p-2, \end{array}$$

where $\lambda = (1-x)$, $\Lambda_1 \cong \mathbf{Z}[x]/(x^p - 1)$ *and we view* R_1 *as a quotient of* Λ_1.

The splintering of these genera into isomorphism classes has been analyzed by Reiner [13]. One can, of course, replace R_1, R_2 by ideal classes in these rings and Λ_1 by $E(\mathfrak{a})$ (cf. section 1) where $\mathfrak{a}$ is an ideal class in R_1. There is an additional invariant lying in a quotient of the group of units of a certain finite ring and, if $p \equiv 1 \pmod 4$ a certain quadratic residue character mod p can also appear as an invariant. The precise result is Theorem 7.3 of [13]. We will require only the observation [13, p. 494] that if $p = 2, 3$ there are no further invariants, i.e. each genus of an indecomposable is a single isomorphism class. In the case $p = 5$ already, although the class number of $\mathbf{Q}(e^{2\pi i/m})$ is one for $m = 5, 25$, the 21 genera of indecomposables split up into 40 isomorphism classes. Hence already the further isomorphism invariants mentioned above exert an influence.

§ 2. Cohomology, restrictions and special classes

If H is a finite group, M an H-lattice then: $H^i(H, M) \cong \oplus_p H^i(H, M_p)$, where p ranges over the primes dividing the order of H [3, p. 84]. Hence if M and M' are locally isomorphic, $H^i(H, M) \cong H^i(H, M')$; so the cohomology of an H-lattice depends only on its genus.

We recall the cohomology of a cyclic group $\mathbf{Z}/n = <\sigma>$ [3, p. 58]. We write $N = 1 + \sigma \ldots + \sigma^{n-1}$ and $D = 1 - \sigma$. If M is a $\mathbf{Z}/n$-module, then

$$H^0(\mathbf{Z}/n, M) = M^\sigma$$
$$H^{2i-1}(\mathbf{Z}/n, M) = {}_N M/D \cdot M$$
$$H^{2i}(\mathbf{Z}/n, M) = M^\sigma/N \cdot M$$

for all $i \geqslant 1$, where M^σ denotes σ-invariants and ${}_N M = \{x \in M : Nx = 0\}$. From these remarks it is easy to compute the cohomology of the indecomposable $\mathbf{Z}/p$-lattices described in section 1.

(2.1) Proposition. *The following table describes the cohomology of the indecomposable $\mathbf{Z}/p$-lattices:*

M	rank	H^0	H^1	H^2
$\mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p$
α	$p-1$	0	$\mathbf{Z}/p$	0
β	p	$\mathbf{Z}$	0	0

Similarly one can easily compute the cohomology and restriction of the first four $\mathbf{Z}/p^2$-lattices of (1.3).

(2.2) PROPOSITION. *If* $[M]_p$ *denotes the restriction of the* $\mathbf{Z}/p^2$*-lattice* M *to the subgroup of order* p, *we have the following table.*

M	rank	H^0	H^1	H^2	$[M]_p$
$M_1 = \mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p^2$	$\mathbf{1}$
$M_2 = R_1$	$p-1$	0	$\mathbf{Z}/p$	0	$(p-1)\mathbf{1}$
$M_3 = R_2$	p^2-p	0	$\mathbf{Z}/p$	0	$p\alpha$
$M_4 = \Lambda_1$	p	$\mathbf{Z}$	0	$\mathbf{Z}/p$	$p\mathbf{1}$

Furthermore, $\Lambda_2 = \mathbf{Z}[\mathbf{Z}/p^2]$, the regular representation of $\mathbf{Z}/p^2$, satisfies $H^0 = \mathbf{Z}$, $H^1 = 0 = H^2$ and $[\Lambda_2]_\beta = p\beta$.

Proof. It suffices to observe that $M_2 = p^*(\alpha)$ and $M_4 = p^*(\beta)$, where $p\colon \mathbf{Z}/p^2 \to \mathbf{Z}/p$ is the natural projection, and M_3 fits into a short exact sequence:

$$0 \to M_4 \to \Lambda_2 \to M_3 \to 0\,.$$

The last remark follows from the freeness of Λ_2.

To complete the table for the modules M_i, $i \geqslant 5$, we have the following lemma:

(2.3) LEMMA. *If* L *is a* $\mathbf{Z}/p$*-lattice,* $\alpha \in L$, *then the extension* $M = (L, \alpha)$ *defined by (1.2) satisfies:*

$$H^2(C; M) \cong \operatorname{coker}\big(x_*\colon H^2(C; \varphi_2\Lambda_2) \to H^2(C; L)\big)$$

where C *is either* $\mathbf{Z}/p^2$ *or* $\mathbf{Z}/p$.

Proof. The diagram (1.2) induces

$$\begin{array}{ccccccccc}
\to & H^1(C, R_2) & \xrightarrow{\delta} & H^2(C, \varphi_2\Lambda_2) & \to & 0 & & \\
 & \downarrow & & \downarrow x_* & & & & \\
\to & H^1(C, R_2) & \xrightarrow[\delta]{} & H^2(C, L) & \to & H^2(C, M) & \to & 0
\end{array}$$

where the zeros follow from (2.2). An easy diagram-chase completes the proof.

(2.4) PROPOSITION. *The following table describes the cohomology of the indecomposable* $\mathbf{Z}/p^2$*-lattices* $M_i, i \geqslant 5$:

M	rank	H^0	H^1	H^2
M_5	$p^2 - p + 1$	$\mathbf{Z}$	0	$\mathbf{Z}/p$
$M_6(0)$	p^2	$\mathbf{Z}$	0	0
$M_6(k)$	p^2	$\mathbf{Z}$	$\mathbf{Z}/p$	$\mathbf{Z}/p$
$M_7(k)$	$p^2 + 1$	$\mathbf{Z} \oplus \mathbf{Z}$	0	$\mathbf{Z}/p \oplus \mathbf{Z}/p$
$M_8(0)$	$p^2 - 1$	0	$\mathbf{Z}/p^2$	0
$M_8(k)$	$p^2 - 1$	0	$\mathbf{Z}/p \oplus \mathbf{Z}/p$	0
$M_9(k)$	p^2	$\mathbf{Z}$	$\mathbf{Z}/p$	$\mathbf{Z}/p$

Proof. Since

$$H^0(\mathbf{Z}/p^2; R_2) = 0, H^0(\mathbf{Z}/p^2; (L, x)) \cong H^0(\mathbf{Z}/p^2; L)$$

and these can be read off from (2.2). The groups H^2 are computed by (2.3). We work out one example in detail. Consider $M_6(k), 0 \leqslant k \leqslant p - 1$, so that $L = \Lambda_1$. If we identify $\varphi_2\Lambda_2$ with Λ_1 then the generator:

$$1 + x + \ldots + x^{p-1} \in H^2(\mathbf{Z}/p^2, \varphi_2\Lambda_2)$$

is sent by λ_*^k, $1 \leqslant k \leqslant p - 1$, to

$$(1-x)^k(1+x+\ldots+x^{p-1}) = (1-x)^{k-1} \cdot 0 = 0$$

in $H^2(\mathbf{Z}/p^2; \Lambda_1)$. If $k = 0$, then the map is an isomorphism. Hence $H^2(\mathbf{Z}/p^2; M_6(0)) = 0$ and $H^2(\mathbf{Z}/p^2, M_6(k)) = \mathbf{Z}/p, k \geqslant 1$.

The groups $H^1(M_i)$ can be read off the long exact cohomology sequence of the bottom row of (1.2).

Remark. It follows from (2.2) that $M_6(0)$ is the genus of the regular representation.

We now record the restrictions of the modules M_i to the subgroup of order p.

(2.5) PROPOSITION. *The* $\mathbf{Z}/p$*-cohomology and the restrictions of* M_i, $i \geqslant 5$, *are given by*:

M	$H^2(\mathbf{Z}/p; [M]_p)$	$[M]_p$	
M_5	0	$(p-1)\alpha + \beta$	
$M_6(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + k\alpha + (p-k)\beta$	$0 \leqslant k \leqslant p-1$
$M_7(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + k\alpha + (p-k)\beta$	$1 \leqslant k \leqslant p-2$
$M_8(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leqslant k \leqslant p-2$
$M_9(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leqslant k \leqslant p-2$

Proof. One begins by computing $H^2(\mathbf{Z}/p, [M]_p)$, from (2.3). We work out an example again with $M = M_6(k)$. We will need these details later. The map

$$p(\mathbf{Z}/p) = H^2(\mathbf{Z}/p, \Lambda_1) \overset{\lambda^k_*}{\to} H^2(\mathbf{Z}/p; \Lambda_1) = p(\mathbf{Z}/p)$$

sends the generator x^j, $0 \leqslant j \leqslant p-1$, from the left-hand side to $x^j(1-x)^k$. The resulting matrix $C_{p,k}$ in $GL_p(\mathbf{Z}/p)$ can be described in the following way. If $p > k$, let $C_{p,k,j}$ denote a column p-vector whose entries are the coefficients of $(1-x)^k$ introduced "cyclically" starting in row j. For example:

$$C_{5,2,4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

We define $C_{p,k}$ to be the $p \times p$ matrix whose j^{th} column is $C_{p,k,j}$. So, for example,

$$C_{5,2} = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

It is a consequence of the identity $(1-x)^{k+1} = (1-x)(1-x)^k$ that

$$C_{p,k+1} = C_{p,1}C_{p,k}$$

so we get:

$$C_{p,k} = C_{p,1}^{k} = \begin{bmatrix} 1 & & & & & -1 \\ -1 & 1 & & & 0 & \\ & -1 & 1 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & -1 & 1 \end{bmatrix}^{k}.$$

It is easy to see that $C_{p,1}(p(\mathbf{Z}/p)) = \{\bar{x} \in p(\mathbf{Z}/p) : \sum_{i=0}^{p-1} x_i = 0\}$. Hence rank $(C_{p,k}) = p - k$ and $\dim_{\mathbf{Z}/p} H^2(\mathbf{Z}/p; M_6(k)) = p - (p-k) = k$, as claimed. The other cases are similar.

Now from (2.1) we see that the $\mathbf{Z}/p$-dimension of H^2 is the multiplicity of $\mathbf{1}$ in $[M]_p$. The multiplicities of α and β are then determined by the bottom row of (1.2) restricted to the subgroup of order p.

Recall from Charlap [4] that a cohomology class $\alpha \in H^2(H, M)$ is called *special* if $i*(\alpha) \neq 0$ for the inclusion $i: C \to H$ of any cyclic subgroup. The basic result is (see [4, p. 22]).

(2.6) PROPOSITION. *The extension* Γ *in* (0, 1) *corresponding to* $\alpha \in H^2(H, M)$ *is torsion-free (i.e. the fundamental group of a flat manifold) if and only if* α *is special.*

It remains to determine which indecomposables in (1.3) admit special classes. The result is:

(2.7) PROPOSITION. *There are* $2p - 1$ *genera of indecomposable* $\mathbf{Z}/p^2$-*lattices that admit special classes. They are* M_1, M_4, $M_7(k)$, $1 \leqslant k \leqslant p-2$, *and* $M_9(k)$, $0 \leqslant k \leqslant p-2$.

Proof. From (2.2), (2.4) and (2.5) one sees that the given lattices along with $M_6(k)$, $1 \leqslant k \leqslant p-2$, are the only possibilities. We must determine the restriction map

$$i*(M): H^2(\mathbf{Z}/p^2; M) \to H^2(\mathbf{Z}/p; M)$$

in these cases. Clearly for $M = M_1$, $i*(M)$ is the natural projection and for $M = M_4$, $i*(M)$ is the diagonal embedding.

Now suppose $M = M_6(k)$. We have a commutative diagram of exact sequences from (1.2) and (2.3)

$$\begin{array}{ccccccc} & & H^2(\mathbf{Z}/p^2; \Lambda_1) & \to & H^2(\mathbf{Z}/p^2; M) & \to & 0 \\ & & \downarrow {\scriptstyle \Delta = i_*(\Lambda_1)} & & \downarrow {\scriptstyle i_*(M)} & & \\ H^2(\mathbf{Z}/p; \varphi_2\Lambda_2) & \overset{\lambda_*^k}{\to} & H^2(\mathbf{Z}/p; \Lambda_1) & \to & H^2(\mathbf{Z}/p; M) & \to & 0 \end{array}$$

where Δ is the diagonal map $\mathbf{Z}_p \to p(\mathbf{Z}/p)$. Hence to eliminate $M_6(k)$, it suffices to show $\mathrm{Im}(\Delta) \subset \mathrm{Im}(\lambda_*^k)$. Let e denote a column p-vector consisting of all 1's, according to the proof of (2.5) we must find an $\bar{x}_k$, $1 \leqslant k \leqslant p-1$ so that $C_{p,k} \cdot \bar{x}_k = C_{p,1}^k \cdot \bar{x}_k = e$. We do this inductively on k. For example, $\bar{x}_1 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ p \end{bmatrix}$, as can easily be checked. Inductively we define

$$\bar{x}_k(i) = \begin{cases} 1 & i = 1 \\ \bar{x}_k(i-1) + \bar{x}_{k-1}(i) & i > 1 . \end{cases}$$

Clearly $C_{p,1} \cdot \bar{x}_k = \bar{x}_{k-1}$, for all coordinates except possibly the first; we must show $\bar{x}_k(p) \equiv 0 \pmod p$. But a comparison of the $\bar{x}_k$'s with Pascal's triangle convinces one that

$$\bar{x}_k(p) = \binom{p-1+k}{p-1} \equiv \binom{k-1}{p-1}\binom{1}{0} \equiv 0 \pmod p,$$

since $k - 1 < p - 1$.

We leave it for the reader to check that the restriction maps for $M_7(k)$ and $M_9(k)$ are non-trivial.

§ 3. Z/4-MANIFOLDS

In this section, we consider the case $p = 2$. For convenience, we change the notation slightly and write M_7 for $M_6(1)$ and M_i for $M_i(0)$, $i = 6, 8, 9$. According to (2.7), the indecomposable $\mathbf{Z}/4$-lattices that carry special classes are M_1, M_4 and M_9. It is easy to see M_i is faithful if and only if

$i = 3, 5, 6, 7, 8, 9$. Hence if $M = \sum m_i M_i$ is an arbitrary $\mathbf{Z}/4$-lattice then M is a faithful representation carrying a special class if and only if the multiplicities m_i satisfy the inequalities:

$$m_1 + m_4 + m_9 > 0$$

(3.0)

$$m_3 + m_5 + m_6 + m_7 + m_8 + m_9 > 0\,.$$

Since the multiplicities are a complete set of isomorphism invariants in the case $p = 2$ (see section 1) one can use the conditions (3.0) to show:

(3.1) THEOREM. *If $L_n(m)$ denotes the number of isomorphism classes of n-dimensional $\mathbf{Z}/m$-lattices that carry special classes, then:*

$$L_n(4) = \sum_{j=2}^{n-1}\left(a_j - \left[\frac{j}{2}\right] - 1\right) + \sum_{j=[n]_2+2}^{n-2}(a_j - a_{j-1} - 1) + \sum_{j=[n]_4}^{n-4}(a_j - a_{j-2} - a_{j-4} + a_{j-6})$$

where $[k]_p$ denotes the reduction of k modulo p, $[k]$ denote the largest integer $\leqslant k$ and the a_j's are given by

$$P(t) = \sum_{j=0}^{\infty} a_j t^j = \frac{1}{(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)^3}$$

In particular, the number of n-dimensional $\mathbf{Z}/4$-manifolds is at least $L_n(4)$.

Proof. If $Q(t)$ is a power series, let $\operatorname{coef}(n, Q(t))$ denote the coefficient of t^n in $Q(t)$. The number $L_n(4)$ counts the number of ways of writing

$$n = m_1 + m_2 + 2(m_3+m_4) + 3(m_5+m_8) + 4(m_6+m_7+m_9)$$

where the m_i's satisfy (3.0). If $m_1 > 0$ there is a contribution:

$$\sum_{m_1=1}^{n-2} \operatorname{coef}(n-m_1, P(t)) - \left(\left[\frac{n-m_1}{2}\right] + 1\right)$$

where $\left[\frac{n-m_1}{2}\right] + 1$ is the number of ways of expressing $n - m_1$ as a combination of 1's (M_2) and 2's (M_4) (not permitted by (3.0)). Reindexing gives the first term for $L_n(4)$.

Similarly, if $m_1 = 0$, $m_4 > 0$ there is a contribution:

$$\sum_{m_4} \operatorname{coef}(t^{n-2m_4}(1-t)P(t)) - 1$$

where 1 is subtracted to omit choosing m_2 alone. Finally, if $m_1 = m_4 = 0$, we have:

$$\sum_{m_9} \text{coef}(t^{n-4m_9}, (1-t^2)(1-t^4)P(t)).$$

The coefficients of $(1-t)P(t)$ and $(1-t^2-t^4-t^6)P(t)$ are easily expressible in terms of the a_j's and the result follows.

Remark. In order for a $\mathbf{Z}/p$-lattice to carry a special class, the multiplicity of the trivial representation must be non-zero. Topologically this is reflected in the fact that a $\mathbf{Z}/p$-manifold fibers over a circle. This is already false for a 4-dimensional $\mathbf{Z}/4$-manifold as the following example shows.

Example. $L_4(4) = 6$. The multiplicities of the indecomposables in these 4-dimensional $\mathbf{Z}/4$-lattices are given by:

notation of [2]	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
07/02/02	1				1				
12/01/04	1							1	
12/01/02	1	1	1						
07/02/01	2		1						
12/01/03			1	1					
12/01/06									1

where the first column gives the label of these "**Z**-classes" from the table of the four-dimensional crystallographic groups in [2]. In fact, as these tables indicate, there is precisely one $\mathbf{Z}/4$-manifold corresponding to each $\mathbf{Z}/4$-lattice, hence there are exactly 6 4-dimensional $\mathbf{Z}/4$-manifolds.

Remark. Recall that if $p < 23$, the field $\mathbf{Q}(e^{2\pi i/p})$ has class number one. This fact, along with the work of Charlap [4], shows that the number of n-dimensional $\mathbf{Z}/p$-manifolds is exactly $L_n(p)$, $p < 23$. This number is readily computable, as Charlap [4, p. 30] remarks, and the precise formula is:

$$L_n(p) = \sum_{j=p-1}^{n-1} \left(\left[\frac{j}{p-1} \right] - \left\langle \frac{j}{p} \right\rangle + 1 \right) \tag{3.2}$$

where $\langle k \rangle$ denotes the smallest integer $\geqslant k$. In particular, $L_p(p) = 1$, $L_n(p) = 0$, $p > n$, and when $p = 2$

$$L_n(2) = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right) - 1 + \frac{[n]_2}{4} \tag{3.3}$$

using the notation of (3.1).

One can easily construct the following table of values of $L_n(p)$:

n	2	3	4	5	6
p					
2	1	3	5	8	11
3		1	2	3	4
5				1	2

Hence 14 of the 74 4-dimensional flat manifolds have cyclic holonomy $\leqslant 5$. (Furthermore, 26 have holonomy the Klein 4-group.) We describe analogous facts in dimension 5 below.

We let $SH^2(H, M)$ denote the set of special classes in $H^2(H, M)$. If H is a cyclic p-group and $i: \mathbf{Z}_p \hookrightarrow H$ is the inclusion of the subgroup of order p, then

$$SH^2(H, M) = H^2(H, M) - \ker(i*) .$$

If N (resp. Z) denotes the normalizer (resp. the centralizer) of H in $\mathrm{Aut}(M)$, there is an exact sequence (see [15, p. 50])

$$0 \to Z \to N \to \mathrm{Aut}(H) .$$

We conjecture:

Conjecture. If $\mathbf{Z}[e^{2\pi i/p^k}]$ is a unique factorization domain for, $1 \leqslant k \leqslant n$, then N acts transitively on $SH^2(\mathbf{Z}/p^n; M)$ for any H-lattice M.

The case $n = 1$ of the Conjecture follows from Charlap [4]. Class number tables shows that the $n = 2$ case applies to $p = 2, 3, 5$, the $n = 3$ case to $2, 3$ and the $n = 4$ case to $p = 2$. This conjecture implies that the lower bound of (3.1) is exact.

We mention that the multiplicities of the indecomposables in the 5-dimensional $\mathbf{Z}/4$-lattices that admit special classes are given by:

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
	1								1
	1						1		
*						1			
	1	1			1				
*	1	1						1	
	1		1	1					
*	1		2						
*	1	2	1						
*	3							1	
	2				1				
*	2	1	1						
	3		1						
*		1	1	1					
				1	1				
*				1				1	
*		1							1

Those lattices that are starred clearly satisfy the Conjecture.

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Howard Hiller

Department of Mathematics
Columbia University
New York, N. Y. 10027
USA

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