

**Zeitschrift:** L'Enseignement Mathématique  
**Band:** 31 (1985)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** FLAT MANIFOLDS WITH  $\mathbb{Z}/p^2\mathbb{Z}$  HOLONOMY  
**Kapitel:** §1. Genera of  $\mathbb{Z}/p^2\mathbb{Z}$ -lattices  
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**DOI:** <https://doi.org/10.5169/seals-54570>

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The case  $H = \mathbf{Z}/p^2$  differs fundamentally from that of  $H = \mathbf{Z}/p$  in that the number of genera of indecomposable  $\mathbf{Z}/p^2$ -lattices is a function of  $p$  (in fact,  $4p+1$ ) while there are 3 genera of indecomposable  $\mathbf{Z}/p$ -lattices, for any prime  $p$ . Furthermore it is no longer the case that only the trivial representation admits non-trivial (and “special”) two-dimensional cohomology classes. Eventually we restrict to the case  $p = 2$  where there are 9 genera (originally described by Roiter [14] correcting a mistake in Diederichsen [6]) and 3 of these admit special classes. The assumption  $p = 2$  also insures that the genera are identical to the isomorphism classes so there are no further invariants to consider. This follows from work of Reiner [13] and the fact that  $\mathbf{Z}[e^{2\pi i/m}]$  is a unique factorization domain for  $m = 2, 4$ .

As the smallest dimension of a  $\mathbf{Z}/p^2$ -manifold is  $p^2 - p + 1$  (this is a special case of results from [8]) only the case  $p = 2$  produces flat manifolds of dimension 5, the smallest dimension for which one lacks a complete classification. We show that there are at least 16 5-dimensional flat manifolds with holonomy  $\mathbf{Z}/4$  and give a general lower bound for any dimension.

The main ingredient for these results is the work of Heller and Reiner [7] on the integral representation theory of  $\mathbf{Z}/p^2$ , reviewed in section 1. In section 2 we study the cohomology of the indecomposables, compute their restrictions to the subgroup of order  $p$  and identify the “special” classes. Finally in section 3 we restrict to the case  $p = 2$  and study the class of  $\mathbf{Z}/4$ -manifolds.

It is hoped that this example of holonomy classification will succeed in exposing the role played by integral representation theory and cohomology of groups in understanding the structure of flat Riemannian manifolds.

It is a pleasure to thank I. Reiner for helpful correspondence concerning integral representation theory.

## § 1. GENERA OF $\mathbf{Z}/p^2$ -LATTICES

We begin by briefly reviewing the language and philosophy of the integral representation theory of finite groups (see [5], [11]). We then give Heller and Reiner’s description [7] of the genera of  $\mathbf{Z}/p^2$  lattices as extensions.

Suppose  $\Lambda$  is a  $\mathbf{Z}$ -order in a  $\mathbf{Q}$ -algebra  $A$ . A  $\Lambda$ -lattice is a left  $\Lambda$ -module that is also a free abelian group of finite rank. The basic problem

of integral representation theory is the classification of such  $\Lambda$ -lattices,  $\Lambda$  fixed. The  $\mathbf{Z}$ -orders that we will need are group rings of finite groups  $\mathbf{Z}G \subset \mathbf{Q}G$  and rings of algebraic integers  $\mathcal{O}_K$  in an algebraic number field  $K$ . We sometimes refer to a  $\mathbf{Z}G$ -lattice as a  $G$ -lattice.

Let  $\mathbf{Z}_p$  (resp.  $\mathbf{Q}_p$ ) denote the  $p$ -adic completion of  $\mathbf{Z}$  (resp.  $\mathbf{Q}$ ). It is easy to see that  $\Lambda_p = \mathbf{Z}_p \otimes \Lambda$  is a  $\mathbf{Z}_p$ -order in the  $\mathbf{Q}_p$ -algebra  $A_p$ . Furthermore, any  $\Lambda$ -lattice  $M$  yields a  $\Lambda_p$ -lattice  $M_p = \mathbf{Z}_p \otimes M$ . One says that  $M$  and  $M'$  are *locally isomorphic* (or in the same *genus*) if  $M_p \cong M'_p$  as  $\Lambda_p$ -modules for all primes  $p$ .

The classification of  $\Lambda$ -lattices is often attacked by a "local-to-global" approach. By this we mean the solution of the following two problems:

1. (local) Determine a complete set of invariants of the genus of a  $\Lambda$ -lattice.
2. (global) Determine a complete set of invariants of the isomorphism class of  $\Lambda$ -lattice within a fixed genus.

This approach has been very successful and the examples below illustrate it.

We introduce some notation. Let  $\omega$  (resp.  $\zeta$ ) denote a primitive  $p^2$  (resp.  $p^{\text{th}}$ ) root of unity and let  $R_1 = \mathbf{Z}[\zeta]$ ,  $R_2 = \mathbf{Z}[\omega]$ . We also let  $\Lambda_i = \mathbf{Z}[\mathbf{Z}/p^i]$ . The classification of lattices over a ring of algebraic integers, or more generally a Dedekind domain is classic, and is a good example of the local-to-global approach.

(1.1) THEOREM (Steinitz). *If  $R$  is a Dedekind domain, every  $R$ -lattice is a direct sum of non-zero ideals of  $R$ . The genus of an  $R$ -lattice is determined by the number of non-zero ideals occurring, its rank. The isomorphism class of the  $R$ -lattice within the genus is determined by the ideal class of the product of the ideals as an element of the ideal class group of  $R$  (the Steinitz class of the lattice).*

The classification problem for  $\mathbf{Z}/p$ -lattices was solved by Diederichsen [6] and Reiner [10]. Again the local-to-global approach is useful.

If  $\mathfrak{a}$  denotes a non-zero ideal of  $R_1$  let  $E(\mathfrak{a})$  denote the non-split extension of  $\mathfrak{a}$  by the trivial lattice  $\mathbf{Z}$ . The genus of  $\mathfrak{a}$  (resp.  $E(\mathfrak{a})$ ) is denoted  $\alpha$  (resp.  $\beta$ ). Every  $\mathbf{Z}/p$ -lattice  $M$  can be written:

$$M = \mathbf{Z}^{(a)} \oplus \sum_{i=1}^b \mathfrak{a}_i \oplus \sum_{i=1}^c E(\mathfrak{a}'_i).$$

The genus of the  $\mathbf{Z}/p$ -lattice  $M$  is determined by the multiplicities  $a, b, c$  of the three indecomposable genera  $\mathbf{1}, \alpha, \beta$ . The isomorphism class of the lattice within its genus is completely determined by the ideal class  $\prod_{i=1}^b \mathfrak{a}_i \cdot \prod_{i=1}^c \mathfrak{a}'_i$  in the ideal class group of  $R_1$ .

The solution of the (local) classification problem for  $R_2 = \mathbf{Z}[\omega]$  (a Dedekind domain) and  $\Lambda_1 = \mathbf{Z}[\mathbf{Z}/p]$  can be combined to classify genera of  $\mathbf{Z}/p^2$ -lattices. The technique used is essentially homological. If  $M$  is a  $\mathbf{Z}/p^2$ -lattice, we let  $L = \{x \in M : (x^p - 1)M = 0\}$ .  $L$  is a  $\mathbf{Z}/p$ -lattice and fits into a  $\mathbf{Z}/p^2$ -exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

where  $N$  is an  $R_2$ -lattice. Hence one is reduced to classifying extensions of  $R_2$ -lattices by  $\mathbf{Z}/p$ -lattices using homological methods. It is not difficult to show (see [13, p. 478]) that  $\text{Ext}_2(R_2, L) \cong L/pL$ , where  $L$  is an arbitrary  $\mathbf{Z}/p$ -lattice. In fact, if  $\alpha \in L/pL$  then the corresponding extension is given by the pushout diagram:

$$(1.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \varphi_2 \Lambda_2 & \rightarrow & \Lambda_2 & \rightarrow & R_2 \rightarrow 0 \\ & & \alpha \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & L & \rightarrow & M & \rightarrow & R_2 \rightarrow 0 \end{array}$$

where  $\varphi_2$  is the cyclotomic polynomial  $1 + x^p + x^{2p} + \dots + x^{(p-1)p}$  and the map  $\alpha$  is (by abuse of language) the map that sends  $\varphi_2$  to  $\alpha$ . We write  $(L, \alpha)$  for this extension. The final result is:

(1.3) THEOREM (Heller and Reiner [7]). *There are  $4p + 1$  genera of indecomposable  $\mathbf{Z}/p^2$ -lattices given by:*

$$\begin{aligned} M_1 &= \mathbf{Z}, \\ M_2 &= R_1, \\ M_3 &= R_2, \\ M_4 &= \Lambda_1, \\ M_5 &= (\mathbf{Z}, 1), \\ M_6(k) &= (\Lambda_1, \lambda^k), & 0 \leq k \leq p-1, \\ M_7(k) &= (\mathbf{Z} \oplus \Lambda_1, \mathbf{1} \oplus \lambda^k), & 1 \leq k \leq p-2, \\ M_8(k) &= (R_1, \lambda^k), & 0 \leq k \leq p-2, \\ M_9(k) &= (\mathbf{Z} \oplus R_1, \mathbf{1} \oplus \lambda^k), & 0 \leq k \leq p-2, \end{aligned}$$

where  $\lambda = (1-x)$ ,  $\Lambda_1 \cong \mathbf{Z}[x]/(x^p-1)$  and we view  $R_1$  as a quotient of  $\Lambda_1$ .

The splintering of these genera into isomorphism classes has been analyzed by Reiner [13]. One can, of course, replace  $R_1, R_2$  by ideal classes in these rings and  $\Lambda_1$  by  $E(\alpha)$  (cf. section 1) where  $\alpha$  is an ideal class in  $R_1$ . There is an additional invariant lying in a quotient of the group of units of a certain finite ring and, if  $p \equiv 1 \pmod{4}$  a certain quadratic residue character mod  $p$  can also appear as an invariant. The precise result is Theorem 7.3 of [13]. We will require only the observation [13, p. 494] that if  $p = 2, 3$  there are no further invariants, i.e. each genus of an indecomposable is a single isomorphism class. In the case  $p = 5$  already, although the class number of  $\mathbf{Q}(e^{2\pi i/m})$  is one for  $m = 5, 25$ , the 21 genera of indecomposables split up into 40 isomorphism classes. Hence already the further isomorphism invariants mentioned above exert an influence.

§ 2. COHOMOLOGY, RESTRICTIONS AND SPECIAL CLASSES

If  $H$  is a finite group,  $M$  an  $H$ -lattice then:  $H^i(H, M) \cong \bigoplus_p H^i(H, M_p)$ , where  $p$  ranges over the primes dividing the order of  $H$  [3, p. 84]. Hence if  $M$  and  $M'$  are locally isomorphic,  $H^i(H, M) \cong H^i(H, M')$ ; so the cohomology of an  $H$ -lattice depends only on its genus.

We recall the cohomology of a cyclic group  $\mathbf{Z}/n = \langle \sigma \rangle$  [3, p. 58]. We write  $N = 1 + \sigma + \dots + \sigma^{n-1}$  and  $D = 1 - \sigma$ . If  $M$  is a  $\mathbf{Z}/n$ -module, then

$$\begin{aligned} H^0(\mathbf{Z}/n, M) &= M^\sigma \\ H^{2i-1}(\mathbf{Z}/n, M) &= {}_N M / D \cdot M \\ H^{2i}(\mathbf{Z}/n, M) &= M^\sigma / N \cdot M \end{aligned}$$

for all  $i \geq 1$ , where  $M^\sigma$  denotes  $\sigma$ -invariants and  ${}_N M = \{x \in M : Nx = 0\}$ . From these remarks it is easy to compute the cohomology of the indecomposable  $\mathbf{Z}/p$ -lattices described in section 1.

(2.1) PROPOSITION. *The following table describes the cohomology of the indecomposable  $\mathbf{Z}/p$ -lattices:*

$M$	rank	$H^0$	$H^1$	$H^2$
$\mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p$
$\alpha$	$p - 1$	0	$\mathbf{Z}/p$	0
$\beta$	$p$	$\mathbf{Z}$	0	0