

§2. Cohomology, restrictions and special classes

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The splintering of these genera into isomorphism classes has been analyzed by Reiner [13]. One can, of course, replace R_1, R_2 by ideal classes in these rings and Λ_1 by $E(\alpha)$ (cf. section 1) where α is an ideal class in R_1 . There is an additional invariant lying in a quotient of the group of units of a certain finite ring and, if $p \equiv 1 \pmod{4}$ a certain quadratic residue character mod p can also appear as an invariant. The precise result is Theorem 7.3 of [13]. We will require only the observation [13, p. 494] that if $p = 2, 3$ there are no further invariants, i.e. each genus of an indecomposable is a single isomorphism class. In the case $p = 5$ already, although the class number of $\mathbf{Q}(e^{2\pi i/m})$ is one for $m = 5, 25$, the 21 genera of indecomposables split up into 40 isomorphism classes. Hence already the further isomorphism invariants mentioned above exert an influence.

§ 2. COHOMOLOGY, RESTRICTIONS AND SPECIAL CLASSES

If H is a finite group, M an H -lattice then: $H^i(H, M) \cong \bigoplus_p H^i(H, M_p)$, where p ranges over the primes dividing the order of H [3, p. 84]. Hence if M and M' are locally isomorphic, $H^i(H, M) \cong H^i(H, M')$; so the cohomology of an H -lattice depends only on its genus.

We recall the cohomology of a cyclic group $\mathbf{Z}/n = \langle \sigma \rangle$ [3, p. 58]. We write $N = 1 + \sigma + \dots + \sigma^{n-1}$ and $D = 1 - \sigma$. If M is a \mathbf{Z}/n -module, then

$$\begin{aligned} H^0(\mathbf{Z}/n, M) &= M^\sigma \\ H^{2i-1}(\mathbf{Z}/n, M) &= {}_N M / D \cdot M \\ H^{2i}(\mathbf{Z}/n, M) &= M^\sigma / N \cdot M \end{aligned}$$

for all $i \geq 1$, where M^σ denotes σ -invariants and ${}_N M = \{x \in M : Nx = 0\}$. From these remarks it is easy to compute the cohomology of the indecomposable \mathbf{Z}/p -lattices described in section 1.

(2.1) PROPOSITION. *The following table describes the cohomology of the indecomposable \mathbf{Z}/p -lattices:*

M	rank	H^0	H^1	H^2
$\mathbf{1}$	1	\mathbf{Z}	0	\mathbf{Z}/p
α	$p - 1$	0	\mathbf{Z}/p	0
β	p	\mathbf{Z}	0	0

Similarly one can easily compute the cohomology and restriction of the first four \mathbf{Z}/p^2 -lattices of (1.3).

(2.2) PROPOSITION. *If $[M]_p$ denotes the restriction of the \mathbf{Z}/p^2 -lattice M to the subgroup of order p , we have the following table.*

M	rank	H^0	H^1	H^2	$[M]_p$
$M_1 = \mathbf{1}$	1	\mathbf{Z}	0	\mathbf{Z}/p^2	$\mathbf{1}$
$M_2 = R_1$	$p - 1$	0	\mathbf{Z}/p	0	$(p-1)\mathbf{1}$
$M_3 = R_2$	$p^2 - p$	0	\mathbf{Z}/p	0	$p\alpha$
$M_4 = \Lambda_1$	p	\mathbf{Z}	0	\mathbf{Z}/p	$p\mathbf{1}$

Furthermore, $\Lambda_2 = \mathbf{Z}[\mathbf{Z}/p^2]$, the regular representation of \mathbf{Z}/p^2 , satisfies $H^0 = \mathbf{Z}$, $H^1 = 0 = H^2$ and $[\Lambda_2]_p = p\beta$.

Proof. It suffices to observe that $M_2 = p^*(\alpha)$ and $M_4 = p^*(\beta)$, where $p: \mathbf{Z}/p^2 \rightarrow \mathbf{Z}/p$ is the natural projection, and M_3 fits into a short exact sequence:

$$0 \rightarrow M_4 \rightarrow \Lambda_2 \rightarrow M_3 \rightarrow 0.$$

The last remark follows from the freeness of Λ_2 .

To complete the table for the modules M_i , $i \geq 5$, we have the following lemma:

(2.3) LEMMA. *If L is a \mathbf{Z}/p -lattice, $\alpha \in L$, then the extension $M = (L, \alpha)$ defined by (1.2) satisfies:*

$$H^2(C; M) \cong \text{coker}(x_*: H^2(C; \varphi_2\Lambda_2) \rightarrow H^2(C; L)).$$

where C is either \mathbf{Z}/p^2 or \mathbf{Z}/p .

Proof. The diagram (1.2) induces

$$\begin{array}{ccccccc} \rightarrow & H^1(C, R_2) & \xrightarrow{\delta} & H^2(C, \varphi_2\Lambda_2) & \rightarrow & 0 & \\ & \downarrow & & \downarrow x_* & & & \\ \rightarrow & H^1(C, R_2) & \xrightarrow{\delta} & H^2(C, L) & \rightarrow & H^2(C, M) & \rightarrow 0 \end{array}$$

where the zeros follow from (2.2). An easy diagram-chase completes the proof.

(2.4) PROPOSITION. *The following table describes the cohomology of the indecomposable \mathbf{Z}/p^2 -lattices $M_i, i \geq 5$:*

M	rank	H^0	H^1	H^2
M_5	$p^2 - p + 1$	\mathbf{Z}	0	\mathbf{Z}/p
$M_6(0)$	p^2	\mathbf{Z}	0	0
$M_6(k)$	p^2	\mathbf{Z}	\mathbf{Z}/p	\mathbf{Z}/p
$M_7(k)$	$p^2 + 1$	$\mathbf{Z} \oplus \mathbf{Z}$	0	$\mathbf{Z}/p \oplus \mathbf{Z}/p$
$M_8(0)$	$p^2 - 1$	0	\mathbf{Z}/p^2	0
$M_8(k)$	$p^2 - 1$	0	$\mathbf{Z}/p \oplus \mathbf{Z}/p$	0
$M_9(k)$	p^2	\mathbf{Z}	\mathbf{Z}/p	\mathbf{Z}/p

Proof. Since

$$H^0(\mathbf{Z}/p^2; R_2) = 0, H^0(\mathbf{Z}/p^2; (L, x)) \cong H^0(\mathbf{Z}/p^2; L)$$

and these can be read off from (2.2). The groups H^2 are computed by (2.3). We work out one example in detail. Consider $M_6(k), 0 \leq k \leq p - 1$, so that $L = \Lambda_1$. If we identify $\varphi_2\Lambda_2$ with Λ_1 then the generator:

$$1 + x + \dots + x^{p-1} \in H^2(\mathbf{Z}/p^2, \varphi_2\Lambda_2)$$

is sent by $\lambda_*^k, 1 \leq k \leq p - 1$, to

$$(1 - x)^k(1 + x + \dots + x^{p-1}) = (1 - x)^{k-1} \cdot 0 = 0$$

in $H^2(\mathbf{Z}/p^2; \Lambda_1)$. If $k = 0$, then the map is an isomorphism. Hence $H^2(\mathbf{Z}/p^2; M_6(0)) = 0$ and $H^2(\mathbf{Z}/p^2, M_6(k)) = \mathbf{Z}/p, k \geq 1$.

The groups $H^1(M_i)$ can be read off the long exact cohomology sequence of the bottom row of (1.2).

Remark. It follows from (2.2) that $M_6(0)$ is the genus of the regular representation.

We now record the restrictions of the modules M_i to the subgroup of order p .

(2.5) PROPOSITION. *The \mathbf{Z}/p -cohomology and the restrictions of $M_i, i \geq 5$, are given by:*

M	$H^2(\mathbf{Z}/p; [M]_p)$	$[M]_p$	
M_5	0	$(p-1)\alpha + \beta$	
$M_6(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + k\alpha + (p-k)\beta$	$0 \leq k \leq p-1$
$M_7(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + k\alpha + (p-k)\beta$	$1 \leq k \leq p-2$
$M_8(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leq k \leq p-2$
$M_9(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leq k \leq p-2$

Proof. One begins by computing $H^2(\mathbf{Z}/p, [M]_p)$, from (2.3). We work out an example again with $M = M_6(k)$. We will need these details later. The map

$$p(\mathbf{Z}/p) = H^2(\mathbf{Z}/p, \Lambda_1) \xrightarrow{\lambda^k} H^2(\mathbf{Z}/p; \Lambda_1) = p(\mathbf{Z}/p)$$

sends the generator $x^j, 0 \leq j \leq p-1$, from the left-hand side to $x^j(1-x)^k$. The resulting matrix $C_{p,k}$ in $GL_p(\mathbf{Z}/p)$ can be described in the following way. If $p > k$, let $C_{p,k,j}$ denote a column p -vector whose entries are the coefficients of $(1-x)^k$ introduced "cyclically" starting in row j . For example:

$$C_{5,2,4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \quad \text{We define } C_{p,k} \text{ to be the } p \times p$$

matrix whose j^{th} column is $C_{p,k,j}$. So, for example,

$$C_{5,2} = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

It is a consequence of the identity $(1-x)^{k+1} = (1-x)(1-x)^k$ that

$$C_{p,k+1} = C_{p,1}C_{p,k}$$

so we get:

$$\begin{array}{ccccccc}
 H^2(\mathbf{Z}/p^2; \Lambda_1) & \rightarrow & H^2(\mathbf{Z}/p^2; M) & \rightarrow & 0 \\
 \downarrow \Delta = i_*(\Lambda_1) & & \downarrow i_*(M) & & \\
 H^2(\mathbf{Z}/p; \varphi_2 \Lambda_2) & \xrightarrow{\lambda_*^k} & H^2(\mathbf{Z}/p; \Lambda_1) & \rightarrow & H^2(\mathbf{Z}/p; M) & \rightarrow & 0
 \end{array}$$

where Δ is the diagonal map $\mathbf{Z}_p \rightarrow p(\mathbf{Z}/p)$. Hence to eliminate $M_6(k)$, it suffices to show $\text{Im}(\Delta) \subset \text{Im}(\lambda_*^k)$. Let e denote a column p -vector consisting of all 1's, according to the proof of (2.5) we must find an $\bar{x}_k, 1 \leq k \leq p - 1$ so that $C_{p,k} \cdot \bar{x}_k = C_{p,1}^k \cdot \bar{x}_k = e$. We do this inductively

on k . For example, $\bar{x}_1 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ p \end{bmatrix}$, as can easily be checked. Inductively we

define

$$\bar{x}_k(i) = \begin{cases} 1 & i = 1 \\ \bar{x}_k(i-1) + \bar{x}_{k-1}(i) & i > 1. \end{cases}$$

Clearly $C_{p,1} \cdot \bar{x}_k = \bar{x}_{k-1}$, for all coordinates except possibly the first; we must show $\bar{x}_k(p) \equiv 0 \pmod{p}$. But a comparison of the \bar{x}_k 's with Pascal's triangle convinces one that

$$\bar{x}_k(p) = \binom{p-1+k}{p-1} \equiv \binom{k-1}{p-1} \binom{1}{0} \equiv 0 \pmod{p},$$

since $k - 1 < p - 1$.

We leave it for the reader to check that the restriction maps for $M_7(k)$ and $M_9(k)$ are non-trivial.

§ 3. $\mathbf{Z}/4$ -MANIFOLDS

In this section, we consider the case $p = 2$. For convenience, we change the notation slightly and write M_7 for $M_6(1)$ and M_i for $M_i(0), i = 6, 8, 9$. According to (2.7), the indecomposable $\mathbf{Z}/4$ -lattices that carry special classes are M_1, M_4 and M_9 . It is easy to see M_i is faithful if and only if